Chapter 6: Nameless Representation of Terms

Terms and Contexts

Shifting and Substitution

Bound Variables

- Recall: bound variables can be renamed, at any moment, to enable substitution:

$[x \mapsto s]x$	$=$	s	
$[x \mapsto s]y$	$=$	y	if $y \neq x$
$[x \mapsto s](\lambda y.t_1)$	$=$	$\lambda y.\ [x \mapsto s]t_1$	if $y \neq x$ and $y \notin FV(s)$
$[x \mapsto s](t_1\ t_2)$	$=$	$[x \mapsto s]t_1\ [x \mapsto s]t_2$	

- Variable Representation
  - Represent variables symbolically, with variable renaming mechanism
  - Represent variables symbolically, with bound variables are all different
  - "Canonically" represent variables in a way such that renaming is unnecessary
  - No use of variables

Terms and Contexts

Nameless Terms

- **De Bruijin** Idea: Replacing named variables by natural numbers, where the number k stands for "the variable bound by the k'th enclosing λ".
  - Examples:

$\lambda x.x$	$\lambda.0$
$\lambda x.\lambda y.\ x\ (y\ x)$	$\lambda.\lambda.\ 1\ (0\ 1)$.

- **Definition** [Terms]: Let T be the smallest family of sets $\{T_0, T_1, T_2, \ldots\}$ such that

1. $k \in T_n$ whenever $0 \le k < n$;
2. if $t_1 \in T_n$ and $n>0$, then $\lambda.t_1 \in T_{n-1}$;
3. if $t_1 \in Tn$ and $t_2 \in T_n$, then $(t_1\ t_2) \in T_n$.

Note: T_n are set of terms with at most n free variables, numbered between 0 and n-1.

Name Context

- Naming Context
  - To deal with terms containing free variables
  - $\Gamma = x \to 4;\ y \to 3;\ z \to 2;\ a \to 1;\ b \to 0$

- Examples

  Under the naming context Γ, we have

  - $x\ (y\ z)$ $\quad$ $4\ (3\ 2)$
  - $\lambda w.\ y\ w$ $\quad$ $\lambda.\ 4\ 0$
  - $\lambda w.\lambda a.x$ $\quad$ $\lambda.\lambda.6$

Shifting and Subtitution

How to define substitution [k → s]t?

Shifting

- Under the naming context x → 1, z → 2
  [1 → 2 (λ.0)] λ.2 → ?
  i.e., [x → z (λw.w)] λy.x → ?

DEFINITION [SHIFTING]: The d-place shift of a term t above cutoff c, written $\uparrow_c^d(\texttt{t})$, is defined as follows:

$$
\begin{aligned}
\uparrow_c^d(\texttt{k}) &= \begin{cases} \texttt{k} & \text{if } k < c \\ \texttt{k}+d & \text{if } k \geq c \end{cases} \\
\uparrow_c^d(\lambda.\texttt{t}_1) &= \lambda.\ \uparrow_{c+1}^d(\texttt{t}_1) \\
\uparrow_c^d(\texttt{t}_1\ \texttt{t}_2) &= \uparrow_c^d(\texttt{t}_1)\ \uparrow_c^d(\texttt{t}_2)
\end{aligned}
$$

We write $\uparrow^d(\texttt{t})$ for $\uparrow_0^d(\texttt{t})$. □

1. What is $\uparrow^2(\lambda.\lambda.\ 1\ (0\ 2))$?
2. What is $\uparrow^2(\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2))$?

Substitution

- Definition

$$[j \mapsto s]k = \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases}$$

$$[j \mapsto s](\lambda.t_1) = \lambda.\ [j+1 \mapsto \uparrow^1(s)]t_1$$

$$[j \mapsto s](t_1\ t_2) = ([j \mapsto s]t_1\ [j \mapsto s]t_2)$$

- Example

  [1 → 2 (λ.0)] λ.2 ➔ λ.3 (λ.0)

  i.e., [x → z (λw.w)] λy.x ➔ λy. z (λw.w)

Evaluation

$$(\lambda x.\ t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

$$(\lambda.t_{12})\ v_2 \longrightarrow\ \uparrow^{-1}([0 \mapsto\ \uparrow^{1}(v_2)]t_{12})$$

Example:

$$(\lambda.1\ 0\ 2)\ (\lambda.0) \longrightarrow 0\ (\lambda.0)\ 1$$

Homework

- Read Chapter 6.
- Do Exercise 6.2.5.