Chapter 5: The Untyped Lambda Calculus

What is lambda calculus for?

Basics: syntax and operational semantics

Programming in the Lambda Calculus

Formalities (formal definitions)

What is Lambda calculus for?

- A core calculus (used by Landin) for
  - capturing the language's essential mechanisms,
  - with a collection of convenient derived forms whose behavior is understood by translating them into the core

- A formal system invented in the 1920s by Alonzo Church (1936, 1941), in which all computation is reduced to the basic operations of function definition and application.

Basics

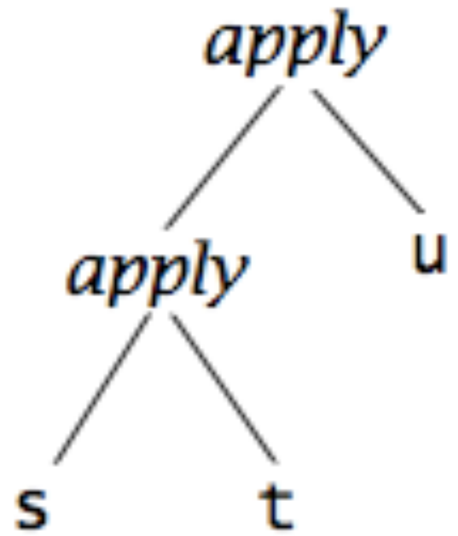
Syntax

- The lambda-calculus (or λ-calculus) embodies this kind of function definition and application in the purest possible form.

t	::=		*terms:*
		x	*variable*
		λx.t	*abstraction*
		t t	*application*

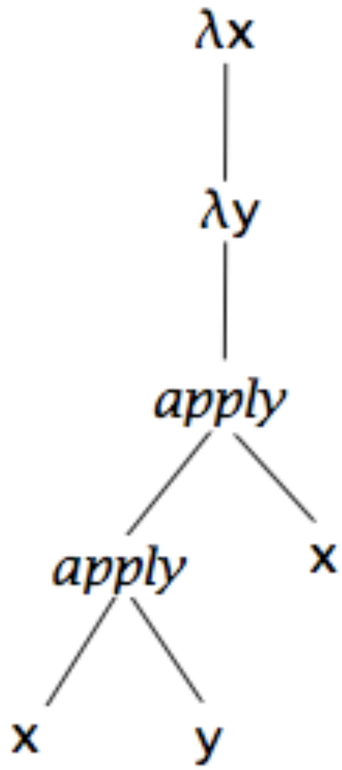
Abstract Syntax Trees

- (s t) u   (or simply written as s t u)



Abstract Syntax Trees

- $\lambda x. (\lambda y. ((x\ y)\ x))$
  (or simply written as $\lambda x.\ \lambda y.\ x\ y\ x$)



Scope

- An occurrence of the variable x is said to be bound when it occurs in the body t of an abstraction λx.t.
  - λx is a binder whose scope is t. A binder can be renamed: e.g., λx.x = λy.y.

- An occurrence of x is free if it appears in a position where it is not bound by an enclosing abstraction on x.
  - **Exercises**: Find free variable occurrences from the following terms: x y, λx.x, λy. x y, (λx.x) x.

Operational Semantics

- Beta-reduction: the only computation

$$(\lambda x.\ t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

"the term obtained by replacing all free occurrences of x in t_{12} by t_2 "

A term of the form (λx.t12) t2 is called a redex.

- Examples:

(λx.x) y → y

(λx. x (λx.x)) (u r) → u r (λx.x)

Evaluation Strategies

- Full beta-reduction
  - Any redex may be reduced at any time.

- Example:
  - Let id = λx.x. We can apply beta reduction to any of the following underlined redexes:

$$\underline{\text{id (id (}\lambda\text{z. id z))}}$$

$$\text{id ((}\underline{\text{id (}\lambda\text{z. id z))}}\text{)}$$

$$\text{id (id (}\lambda\text{z. }\underline{\text{id z}}\text{))}$$

Evaluation Strategies

- The normal order strategy
  - The leftmost, outmost redex is always reduced first.

$$
\begin{array}{ll}
 & \underline{\text{id}\ (\text{id}\ (\lambda z.\ \text{id}\ z))} \\
\longrightarrow & \underline{\text{id}\ (\lambda z.\ \text{id}\ z)} \\
\longrightarrow & \lambda z.\ \underline{\text{id}\ z} \\
\longrightarrow & \lambda z. z \\
\not\longrightarrow &
\end{array}
$$

Evaluation Strategies

- The call-by-name strategy
  - A more restrictive normal order strategy, allowing no reduction inside abstraction.

$$
\begin{array}{ll}
 & \underline{\text{id}\ (\text{id}\ (\lambda z.\ \text{id}\ z))} \\
\longrightarrow & \underline{\text{id}\ (\lambda z.\ \text{id}\ z)} \\
\longrightarrow & \lambda z.\ \text{id}\ z \\
\not\longrightarrow &
\end{array}
$$

Evaluation Strategies

- The call-by-value strategy
  - only outermost redexes are reduced and where a redex is reduced only when its right-hand side has already been reduced to a value, a term that cannot be reduced any more.

$$
\begin{array}{ll}
 & \mathsf{id}\ \underline{(\mathsf{id}\ (\lambda z.\ \mathsf{id}\ z))} \\
\longrightarrow & \underline{\mathsf{id}\ (\lambda z.\ \mathsf{id}\ z)} \\
\longrightarrow & \lambda z.\ \mathsf{id}\ z \\
\not\longrightarrow &
\end{array}
$$

Programming in the Lambda Calculus

Multiple Arguments

Church Booleans

Pairs

Church Numerals

Recursion

Multiple Arguments

$$f\,(x,\ y) = s$$

currying

$$(f\ x)\ y = s$$

$$f = \lambda x.\ (\lambda y.\ s)$$

Church Booleans

- Boolean values can be encoded as:

$$tru = \lambda t.\ \lambda f.\ t$$
$$fls = \lambda t.\ \lambda f.\ f$$

- Boolean conditional and operators can be encoded as:

$$test = \lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n$$
$$and = \lambda b.\ \lambda c.\ b\ c\ fls$$

Church Booleans

- An Example

$$
\begin{array}{ll}
 & \texttt{test tru v w} \\
= & \underline{(\lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n)\ \texttt{tru}}\ v\ w \\
\longrightarrow & \underline{(\lambda m.\ \lambda n.\ \texttt{tru}\ m\ n)\ v}\ w \\
\longrightarrow & \underline{(\lambda n.\ \texttt{tru}\ v\ n)\ w} \\
\longrightarrow & \texttt{tru}\ v\ w \\
= & \underline{(\lambda t.\lambda f.t)\ v}\ w \\
\longrightarrow & \underline{(\lambda f.\ v)\ w} \\
\longrightarrow & v
\end{array}
$$

Church Numerals

- Encoding Church numerals:

```
c0 = λs. λz. z;
c1 = λs. λz. s z;
c2 = λs. λz. s (s z);
c3 = λs. λz. s (s (s z));
etc.
```

- Defining functions on Church numerals:

```
succ = λn. λs. λz. s (n s z);
plus = λm. λn. λs. λz. m s (n s z);
times = λm. λn. m (plus n) c0;
```

Pairs

- Encoding

```
pair = λf.λs.λb. b f s;
fst = λp. p tru;
snd = λp. p fls;
```

- An Example

$$
\begin{aligned}
& \texttt{fst (pair v w)} \\
= \;& \texttt{fst (}\underline{\texttt{(λf. λs. λb. b f s) v}}\texttt{ w)} \\
\longrightarrow \;& \texttt{fst (}\underline{\texttt{(λs. λb. b v s) w}}\texttt{)} \\
\longrightarrow \;& \texttt{fst (λb. b v w)} \\
= \;& \underline{\texttt{(λp. p tru) (λb. b v w)}} \\
\longrightarrow \;& \underline{\texttt{(λb. b v w) tru}} \\
\longrightarrow \;& \texttt{tru v w} \\
\longrightarrow^{*} \;& \texttt{v}
\end{aligned}
$$

Recursion

- Terms with no normal form are said to diverge.

  omega = (λx. x x) (λx. x x);

- Fixed-point combinator

  fix = λf. (λx. f (λy. x x y)) (λx. f (λy. x x y));

  Note: fix f = f (fix f)

Recursion

- Basic Idea:

A recursive definition: h = <body containing h>

$$g = \lambda f \,.\, \langle\text{body containing } f\rangle$$

$$h = \text{fix } g$$

Recursion

- Example:

```
fac = λn. if eq n c0
            then c1
            else times n (fac (pred n))
```

↓

```
g = λf . λn. if eq n c0
                then c1
                else times n (f (pred n))
fac = fix g
```

Exercise: Check that fac c3 → c6.

Formalities (Formal Definitions)

Syntax (free variables)

Substitution

Operational Semantics

Syntax

- **Definition** [Terms]: Let V be a countable set of variable names. The set of terms is the smallest set T such that
  1. $x \in T$ for every $x \in V$;
  2. if $t_1 \in T$ and $x \in V$, then $\lambda x.t_1 \in T$;
  3. If $t1 \in T$ and $t_2 \in T$, then $t_1\ t_2 \in T$.

- Free Variables

$$FV(x) = \{x\}$$

$$FV(\lambda x.t_1) = FV(t_1) \setminus \{x\}$$

$$FV(t_1\ t_2) = FV(t_1) \cup FV(t_2)$$

Substitution

$$
\begin{aligned}
[x \mapsto s]x &= s & \\
[x \mapsto s]y &= y & \text{if } y \neq x \\
[x \mapsto s](\lambda y.t_1) &= \lambda y.\ [x \mapsto s]t_1 & \text{if } y \neq x \text{ and } y \notin FV(s) \\
[x \mapsto s](t_1\ t_2) &= [x \mapsto s]t_1\ [x \mapsto s]t_2 &
\end{aligned}
$$

Example:

$[x \to y\ z]\ (\lambda\, y.\ x\ y)$

$= [x \to y\ z]\ (\lambda\, w.\ x\ w)$

$= \lambda\, w.\ y\ z\ w$

Operational Semantics

Syntax

t	::=		*terms:*
		x	*variable*
		$\lambda x.t$	*abstraction*
		t t	*application*
v	::=		*values:*
		$\lambda x.t$	*abstraction value*

Evaluation $\boxed{t \longrightarrow t'}$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2} \quad \text{(E-APP1)}$$

$$\frac{t_2 \longrightarrow t_2'}{\underline{v_1}\ t_2 \longrightarrow \underline{v_1}\ t_2'} \quad \text{(E-APP2)}$$

$$(\lambda x.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12} \quad \text{(E-APPABS)}$$

Summary

- What is lambda calculus for?
  - A core calculus for capturing language essential mechanisms
  - Simple but powerful
- Syntax
  - Function definition + function application
  - Binder, scope, free variables
- Operational semantics
  - Substitution
  - Evaluation strategies: normal order, call-by-name, call-by-value

Homework

- Understand Chapter 5.
- Do exercise 5.3.6 in Chapter 5.

5.3.6 EXERCISE [★★]: Adapt these rules to describe the other three strategies for evaluation—full beta-reduction, normal-order, and lazy evaluation. □