

Chapter 6: Nameless Representation of Terms

Terms and Contexts
Shifting and Substitution



Bound Variables

- Recall: bound variables can be renamed, at any moment, to enable substitution:

$$[x \mapsto s]x = s$$

$$[x \mapsto s]y = y \quad \text{if } y \neq x$$

$$[x \mapsto s](\lambda y. t_1) = \lambda y. [x \mapsto s]t_1 \quad \text{if } y \neq x \text{ and } y \notin FV(s)$$

$$[x \mapsto s](t_1 t_2) = [x \mapsto s]t_1 [x \mapsto s]t_2$$

- Variable Representation
 - Represent variables symbolically, with variable renaming mechanism
 - Represent variables symbolically, with bound variables are all different
 - “Canonically” represent variables in a way such that renaming is unnecessary
 - No use of variables





Terms and Contexts



Nameless Terms



- **De Bruijn** Idea: Replacing named variables by natural numbers, where the number k stands for “the variable bound by the k 'th enclosing λ ”.

– Examples:

$\lambda x.x$

$\lambda .0$

$\lambda x.\lambda y. x (y x)$

$\lambda .\lambda . 1 (0 1).$

- **Definition** [Terms]: Let T be the smallest family of sets $\{T_0, T_1, T_2, \dots\}$ such that
 1. $k \in T_n$ whenever $0 \leq k < n$;
 2. if $t_1 \in T_n$ and $n > 0$, then $\lambda .t_1 \in T_{n-1}$;
 3. if $t_1 \in T_n$ and $t_2 \in T_n$, then $(t_1 t_2) \in T_n$.

Note: T_n are set of terms with at most n free variables, numbered between 0 and $n-1$.



Name Context



- Naming Context

- To deal with terms containing free variables
- $\Gamma = x \rightarrow 4; y \rightarrow 3; z \rightarrow 2; a \rightarrow 1; b \rightarrow 0$

- Examples

Under the naming context Γ , we have

- $x (y z)$ $4 (3 2)$
- $\lambda w. y w$ $\lambda. 4 0$
- $\lambda w. \lambda a. x$ $\lambda. \lambda. 6$



Shifting and Substitution

How to define substitution $[k \rightarrow s]t$?



Shifting

- Under the naming context $x \rightarrow 1, z \rightarrow 2$
 $[1 \rightarrow 2 (\lambda . 0)] \lambda . 2 \rightarrow ?$
i.e., $[x \rightarrow z (\lambda w . w)] \lambda y . x \rightarrow ?$

DEFINITION [SHIFTING]: The d -place shift of a term t above cutoff c , written $\uparrow_c^d(t)$, is defined as follows:

$$\begin{aligned}\uparrow_c^d(k) &= \begin{cases} k & \text{if } k < c \\ k + d & \text{if } k \geq c \end{cases} \\ \uparrow_c^d(\lambda . t_1) &= \lambda . \uparrow_{c+1}^d(t_1) \\ \uparrow_c^d(t_1 t_2) &= \uparrow_c^d(t_1) \uparrow_c^d(t_2)\end{aligned}$$

We write $\uparrow^d(t)$ for $\uparrow_0^d(t)$.

□

1. What is $\uparrow^2(\lambda . \lambda . 1 (0 2))$?
2. What is $\uparrow^2(\lambda . 0 1 (\lambda . 0 1 2))$?



Substitution

- Definition

$$\begin{aligned} [j \mapsto s]k &= \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \\ [j \mapsto s](\lambda. t_1) &= \lambda. [j+1 \mapsto \uparrow^1(s)]t_1 \\ [j \mapsto s](t_1 t_2) &= ([j \mapsto s]t_1 [j \mapsto s]t_2) \end{aligned}$$

- Example

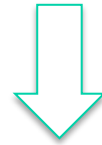
$$[1 \mapsto 2 (\lambda. 0)] \lambda. 2 \rightarrow \lambda. 3 (\lambda. 0)$$

$$\text{i.e., } [x \mapsto z (\lambda w. w)] \lambda y. x \rightarrow \lambda y. z (\lambda w. w)$$



Evaluation

$$(\lambda x. t_{12}) t_2 \rightarrow [x \mapsto t_2]t_{12},$$



$$(\lambda. t_{12}) v_2 \rightarrow \uparrow^{-1}([0 \mapsto \uparrow^1(v_2)]t_{12})$$

Example:

$$(\lambda. 1\ 0\ 2)\ (\lambda. 0) \rightarrow 0\ (\lambda. 0)\ 1$$



Homework



- Read Chapter 6.
- Do Exercise 6.2.5.

6.2.5 EXERCISE [★]: Convert the following uses of substitution to nameless form, assuming the global context is $\Gamma = a, b$, and calculate their results using the above definition. Do the answers correspond to the original definition of substitution on ordinary terms from §5.3?

1. $[b \mapsto a] (b (\lambda x. \lambda y. b))$
2. $[b \mapsto a (\lambda z. a)] (b (\lambda x. b))$
3. $[b \mapsto a] (\lambda b. b a)$
4. $[b \mapsto a] (\lambda a. b a)$

□

