Chapter 6: Nameless Representation of Terms

Terms and Contexts

Shifting and Substitution

Bound Variables

- Recall: bound variables can be renamed, at any moment, to enable substitution:

$$
\begin{array}{lcll}
[x \mapsto s]x & = & s & \\
[x \mapsto s]y & = & y & \text{if } y \neq x \\
\boxed{[x \mapsto s](\lambda y.t_1) \;=\; \lambda y.\ [x \mapsto s]t_1 \quad \text{if } y \neq x \text{ and } y \notin FV(s)} \\
[x \mapsto s](t_1\ t_2) & = & [x \mapsto s]t_1\ [x \mapsto s]t_2 &
\end{array}
$$

- Variable Representation
  - Represent variables symbolically, with variable renaming mechanism
  - Represent variables symbolically, with bound variables are all different
  - "Canonically" represent variables in a way such that renaming is unnecessary
  - No use of variables

Terms and Contexts

Nameless Terms

- **De Bruijin** Idea: Replacing named variables by natural numbers, where the number k stands for "the variable bound by the k'th enclosing λ".
  - Examples:

    $\lambda x.x$ $\qquad\qquad$ $\lambda.0$

    $\lambda x.\lambda y.\ x\ (y\ x)$ $\qquad\qquad$ $\lambda.\lambda.\ 1\ (0\ 1)$.

- **Definition** [Terms]: Let T be the smallest family of sets $\{T_0, T_1, T_2, \ldots\}$ such that

1. $k \in T_n$ whenever $0 \leq k < n$;
2. if $t_1 \in T_n$ and $n>0$, then $\lambda.t_1 \in T_{n-1}$;
3. if $t_1 \in Tn$ and $t_2 \in T_n$, then $(t_1\ t_2) \in T_n$.

Note: T_n are set of terms with at most n free variables, numbered between 0 and n-1.

Name Context

- Naming Context
  - To deal with terms containing free variables
  - $\Gamma = x \to 4;\ y \to 3;\ z \to 2;\ a \to 1;\ b \to 0$

- Examples

  Under the naming context Γ, we have

  - $x\ (y\ z)$ $\quad$ $4\ (3\ 2)$
  - $\lambda w.\ y\ w$ $\quad$ $\lambda.\ 4\ 0$
  - $\lambda w.\lambda a.x$ $\quad$ $\lambda.\lambda.6$

Shifting and Subtitution

How to define substitution $[k \rightarrow s]t$?

Shifting

- Under the naming context x → 1, z → 2
  [1 → 2 (λ.0)] λ.2 → ?
  i.e., [x → z (λw.w)] λy.x → ?

DEFINITION [SHIFTING]: The d-place shift of a term t above cutoff c, written $\uparrow_c^d(\mathtt{t})$, is defined as follows:

$$
\begin{aligned}
\uparrow_c^d(\mathtt{k}) &= \begin{cases} \mathtt{k} & \text{if } k < c \\ \mathtt{k} + d & \text{if } k \geq c \end{cases} \\
\uparrow_c^d(\lambda.\mathtt{t}_1) &= \lambda.\ \uparrow_{c+1}^d(\mathtt{t}_1) \\
\uparrow_c^d(\mathtt{t}_1\ \mathtt{t}_2) &= \uparrow_c^d(\mathtt{t}_1)\ \uparrow_c^d(\mathtt{t}_2)
\end{aligned}
$$

We write $\uparrow^d(\mathtt{t})$ for $\uparrow_0^d(\mathtt{t})$. □

1. What is $\uparrow^2(\lambda.\lambda.\ 1\ (0\ 2))$?

2. What is $\uparrow^2(\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2))$?

Substitution

- Definition

$$
\begin{aligned}
[j \mapsto s]k &= \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases} \\
[j \mapsto s](\lambda.t_1) &= \lambda.\ [j{+}1 \mapsto \uparrow^1(s)]t_1 \\
[j \mapsto s](t_1\ t_2) &= ([j \mapsto s]t_1\ [j \mapsto s]t_2)
\end{aligned}
$$

- Example

  [1 → 2 (λ.0)] λ.2 ➔ λ.3 (λ.0)

  i.e., [x → z (λw.w)] λy.x ➔ λy. z (λw.w)

Evaluation

$$(\lambda x.\ t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

$$(\lambda.t_{12})\ v_2 \longrightarrow \uparrow^{-1}([0 \mapsto \uparrow^{1}(v_2)]t_{12})$$

Example:

$$(\lambda.1\ 0\ 2)\ (\lambda.0) \longrightarrow 0\ (\lambda.0)\ 1$$

Homework

- Read Chapter 6.
- Do Exercise 6.2.5.

6.2.5 EXERCISE [★]: Convert the following uses of substitution to nameless form, assuming the global context is $\Gamma = \mathtt{a},\mathtt{b}$, and calculate their results using the above definition. Do the answers correspond to the original definition of substitution on ordinary terms from §5.3?

1. $[\mathtt{b} \mapsto \mathtt{a}]\ (\mathtt{b}\ (\lambda\mathtt{x}.\lambda\mathtt{y}.\mathtt{b}))$
2. $[\mathtt{b} \mapsto \mathtt{a}\ (\lambda\mathtt{z}.\mathtt{a})]\ (\mathtt{b}\ (\lambda\mathtt{x}.\mathtt{b}))$
3. $[\mathtt{b} \mapsto \mathtt{a}]\ (\lambda\mathtt{b}.\ \mathtt{b}\ \mathtt{a})$
4. $[\mathtt{b} \mapsto \mathtt{a}]\ (\lambda\mathtt{a}.\ \mathtt{b}\ \mathtt{a})$ □