Chapter 13: Reference

Why reference

Typing

Evaluation

Store Typings

Safety

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References

Computational Effects

Also known as *side effects*. A *function* or *expression* is said to have a **side effect** if, in addition to returning a value, it also *modifies some state* or has an *observable interaction with* calling functions or the outside world.

- modify a *global variable* or *static variable*, modify *one of its arguments*,
- *raise an exception,*
- *write* data to *a display* or file, *read* data, or
- call other side-effecting functions.

In the presence of side effects, a program's behavior may depend on *history*; that is, the *order of evaluation* matters.

Computational Effects

Side effects are the *most common way* that a program *interacts with the outside world* (people, file systems, other computers on networks).

The degree to which side effects are used depends on the *programming paradigm*.

- *Imperative programming* is known for *its frequent utilization* of side effects.
- In *functional programming*, side effects are rarely used. Functional languages such as *Standard ML*, *Scheme* and *Scala* do not restrict side effects, but it is customary for programmers to avoid them. The functional language *Haskell* expresses side effects such as I/O and other stateful computations using *monadic* actions.

Mutability

So far, what we have discussed does not yet include *computational effects* (also known as *side effects*) . In particular, whenever we defined function, we *never changed variables or data*. Rather, we always computed new data.

- For instance, the operations to insert an item into the data structure *didn't effect the old copy* of the data structure. Instead, we *always built a new data structure* with the item appropriately inserted.
- For the most part, programming in a functional style (i.e., without side effects) is a "good thing" because it's *easier to reason locally about the behavior* of the program.

Mutability

In most programming languages, *variables are mutable* — i.e., a variable provides both

- a name that refers to a previously calculated value, and
- the possibility of overwriting this value with another (which will be referred to by the same name)

In some languages (e.g., OCaml), these features are separate:

- variables are only for naming — the binding between a variable and its value is immutable
- introduce a new class of mutable values (called *reference cells* or *references*)
- at any given moment, a reference *holds a value* (and can be dereferenced to obtain this value)
- *a new value* may be assigned to a reference

Mutability

Writing values into memory locations is the fundamental mechanism of imperative languages such as Pascal or C.

Mutable structures are required to implement many *efficient algorithms*. They are also very convenient to represent the *current state of a state machine*.

Basic Examples

```
#let r = ref 5
val r : int ref = {contents  = 5}

# r:= !r +2
# !r
-: int = 7

(r:=succ(!r); !r)
(r:=succ(!r); r:=succ(!r); r:=succ(!r);  r:=succ(!r); !r)

 i.e.,
  ((((r:=succ(!r); r:=succ(!r)); r:=succ(!r));  :=succ(!r)); !r)
```

Basic Examples

```
# let flag = ref true;;
-val flag: bool ref = {contents = true}

# if !flag then 1 else 2;;
-: int = 1
```

Reference

Basic operations :

- allocation :   ref (operator)
- dereferencing : !
- assignment: :=

Is there any difference between?

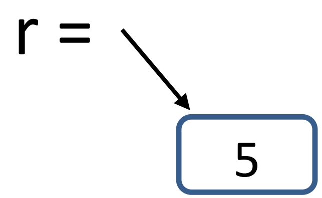
5 + 8;

r: =7;

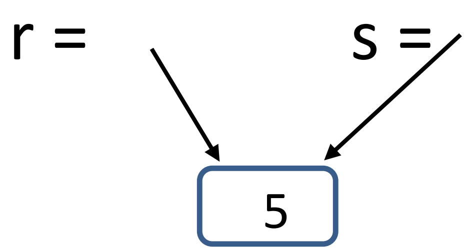
(r:=succ(!r); !r)

Aliasing

A value of type $\mathrm{ref}\ \mathrm{T}$ is a *pointer* to a cell holding a value of type T.



If this value is “copied” by assigning it to another variable, the cell pointed to is not copied. (*r* and *s* are *aliases*)



So we can change r by assigning to s:

(s:=10; !r)

Aliasing all around us

Reference cells are not the only language feature that introduces the possibility of aliasing.

- arrays
- communication channels
- I/O devices (disks, etc.)

The difficulties of aliasing

The possibility of aliasing *invalidates* all sorts of useful forms of *reasoning about programs*, both *by programmers*...

e.g., function

$$\lambda r{:}\,Ref\ Nat.\,\lambda s{:}\,Ref\ Nat.\,(r := 2;\ s := 3;\ !r)$$

always returns 2 unless r and s are aliases.

... and *by compilers*:

Code motion out of loops, common sub-expression elimination, allocation of variables to registers, and detection of uninitialized variables all depend upon the compiler knowing which objects a load or a store operation could reference.

High-performance compilers spend significant energy on *alias analysis* to try to establish when different variables cannot possibly refer to the same storage.

The benefits of aliasing

The problems of aliasing have led some language designers simply to disallow it (e.g., Haskell).

However, there are good reasons why most languages do provide constructs involving aliasing:

- efficiency (e.g., arrays)
- “action at a distance” (e.g., symbol tables)
- shared resources (e.g., locks) in concurrent systems
- ……

Example

$c = ref\ 0$

$\text{incc} = \lambda x{:}\,Unit.\,(c := succ(!\,c);\,!\,c)$

$\text{decc} = \lambda x{:}\,Unit.\,(c := pred(!\,c);\,!\,c)$

$\text{incc}\ unit$

$decc\ unit$

$\text{o} = \{\text{i} = incc, d = decc\}$

$let\ newcounter = o$

$\lambda_{.Unit}.$

$\text{let}\ c\ =\ ref\ 0\ \text{in}$

$\text{let incc} = \lambda x{:}\,Unit.\,(c := succ(!\,c);\,!\,c)\ \text{in}$

$\text{let decc} = \lambda x{:}\,Unit.\,(c := pred(!\,c);\,!\,c)$

$\text{let o} = \{i = incc, d = decc\}\ \text{in}$

o

Syntax

`t ::=`		*terms*
	`unit`	*unit constant*
	`x`	*variable*
	`λx:T.t`	*abstraction*
	`t t`	*application*
	`ref t`	*reference creation*
	`!t`	*dereference*
	`t:=t`	*assignment*

... plus other familiar types, in examples.

Typing rules

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \vdash \mathtt{ref}\ \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1} \qquad \text{(T-Ref)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1}{\Gamma \vdash\ !\mathtt{t}_1 : \mathtt{T}_1} \qquad \text{(T-Deref)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1 \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_1}{\Gamma \vdash \mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 : \mathtt{Unit}} \qquad \text{(T-Assign)}$$

Example

```
NatArray = Ref (Nat→Nat);

newarray = λ_:Unit. ref (λn:Nat.0);
         : Unit → NatArray

lookup = λa:NatArray. λn:Nat. (!a) n;
       : NatArray → Nat → Nat

update = λa:NatArray. λm:Nat. λv:Nat.
           let oldf = !a in
           a := (λn:Nat. if equal m n then v else oldf n);
       : NatArray → Nat → Nat → Unit
```

Evaluation

What is the value of the expression ref 0 ?

Crucial observation: evaluating ref 0 must *do* something ?

Is

r = ref 0

s = ref 0

and

r = ref 0

s = r

behave the same?

Specifically, evaluating ref 0 should *allocate some storage* and yield a *reference* (or *pointer*) to that storage.

So *what* is a reference?

The store

A reference names a *location* in the *store* (also known as the *heap* or just the *memory*).

What is the **store**?

- *Concretely*: an array of *8-bit bytes*, indexed by 32/64-bit integers.
- *More abstractly*: an array of *values*.
- *Even more abstractly*: a *partial function* from locations to values.

Locations

Syntax of *values*:

v	::=		*values*
		unit	*unit constant*
		λx:T.t	*abstraction value*
		l	*store location*

... and since all values are terms ...

Syntax of Terms

t ::=		*terms*
	unit	*unit constant*
	x	*variable*
	λx:T.t	*abstraction*
	t t	*application*
	ref t	*reference creation*
	!t	*dereference*
	t:=t	*assignment*
	$\boxed{l}$	store location

Aside

Does this mean we are going to allow programmers to *write explicit locations* in their programs??

No: This is just a modeling trick.

> We are enriching the "source language" to include some *runtime structures*, so that we can continue to *formalize evaluation* as a relation between source terms.

Aside: If we formalize evaluation in the *big-step style*, then we can *add locations* to *the set of values* (results of evaluation) without adding them to the set of terms.

Evaluation

The *result* of *evaluating a term* now (with references)

- *depends on the store* in which it is evaluated.
- *is not just a value* — we must also keep track of the *changes* that get made to the *store*.

i.e., the evaluation relation should now map *a term as well as a store* to *a reduced term and a new store*.

$$t \mid \mu \;\rightarrow\; t' \mid \mu'$$

To use the metavariable μ to *range over stores*.

Evaluation

An assignment $t_1 \coloneqq t_2$ first evaluates t_1 and t_2 *until they become values* ...

$$\frac{\mathtt{t}_1 \mid \mu \longrightarrow \mathtt{t}_1' \mid \mu'}{\mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_1' \mathtt{:=} \mathtt{t}_2 \mid \mu'} \qquad \text{(E-Assign1)}$$

$$\frac{\mathtt{t}_2 \mid \mu \longrightarrow \mathtt{t}_2' \mid \mu'}{\mathtt{v}_1 \mathtt{:=} \mathtt{t}_2 \mid \mu \longrightarrow \mathtt{v}_1 \mathtt{:=} \mathtt{t}_2' \mid \mu'} \qquad \text{(E-Assign2)}$$

... and then returns unit and updates the store:

$$l \mathtt{:=} \mathtt{v}_2 \mid \mu \longrightarrow \mathtt{unit} \mid [l \mapsto \mathtt{v}_2]\mu \qquad \text{(E-Assign)}$$

Evaluation

A term of *the form* $\text{ref } t_1$ first *evaluates* inside t_1 *until it becomes a value* ...

$$\frac{t_1 \mid \mu \longrightarrow t_1' \mid \mu'}{\texttt{ref } t_1 \mid \mu \longrightarrow \texttt{ref } t_1' \mid \mu'} \qquad \text{(E-REF)}$$

... and then *chooses* (allocates) a *fresh location* l, *augments* the store with a binding from l to v_1, and returns l :

$$\frac{l \notin dom(\mu)}{\texttt{ref } v_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto v_1)} \qquad \text{(E-REFV)}$$

Evaluation

A term $!t_1$ first evaluates in t_1 *until it becomes a value*...

$$\frac{t_1 \mid \mu \longrightarrow t_1' \mid \mu'}{!t_1 \mid \mu \longrightarrow !t_1' \mid \mu'} \qquad \text{(E-Deref)}$$

... and then *looks up this value* (which must be a location, if the original term was well typed) and returns its contents in the current store

$$\frac{\mu(l) = v}{!l \mid \mu \longrightarrow v \mid \mu} \qquad \text{(E-DerefLoc)}$$

Evaluation

Evaluation rules for *function abstraction* and *application* are augmented with stores, but *don't do anything* with them directly.

$$\frac{\mathtt{t}_1 | \mu \longrightarrow \mathtt{t}_1' | \mu'}{\mathtt{t}_1\ \mathtt{t}_2 | \mu \longrightarrow \mathtt{t}_1'\ \mathtt{t}_2 | \mu'} \qquad \text{(E-App1)}$$

$$\frac{\mathtt{t}_2 | \mu \longrightarrow \mathtt{t}_2' | \mu'}{\mathtt{v}_1\ \mathtt{t}_2 | \mu \longrightarrow \mathtt{v}_1\ \mathtt{t}_2' | \mu'} \qquad \text{(E-App2)}$$

$$(\lambda \mathtt{x}:\mathtt{T}_{11}.\mathtt{t}_{12})\ \mathtt{v}_2 | \mu \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} | \mu \qquad \text{(E-AppAbs)}$$

Aside

Garbage Collection

Note that we are not modeling *garbage collection* — the store just *grows without bound*.

It may not be problematic for most *theoretical purposes*, whereas it is clear that for *practical purposes* some form of *deallocation* of unused storage must be provided.

Pointer Arithmetic

p++;

We can’t do any!

Store Typing

Typing Locations

Question: What is the *type of a location*?

Answer: Depends on the contents of the store!

For example,
in the store $(l_1 \mapsto \text{unit}, l_2 \mapsto \text{unit})$, the term $!\, l_2$ is evaluated to unit, having type Unit.

But in the store $(l_1 \mapsto \text{unit}, l_2 \mapsto \lambda x{:}\, \text{Unit}.\, x)$, the term $!\, l_2$ has type $\text{Unit} \rightarrow \text{Unit}$.

Typing Locations — first try

Roughly, to find the type of a location l, first *look up* the current contents of l in the store, and calculate the type T_1 of the contents:

$$\frac{\Gamma \vdash \mu(l) : \mathtt{T}_1}{\Gamma \vdash l : \mathtt{Ref}\ \mathtt{T}_1}$$

More precisely, to make the type of a term depend on the store (keeping a consistent state), we should change the *typing relation* from *three-place* to :

$$\frac{\Gamma \mid \mu \vdash \mu(l) : \mathtt{T}_1}{\Gamma \mid \mu \vdash l : \mathtt{Ref}\ \mathtt{T}_1}$$

i.e., typing is now a *four-place relation* (about *contexts*, *stores*, *terms*, and *types*), though the store is a part of the context ……

Problems #1

However, this rule is not *completely satisfactory*, and is *rather inefficient*.

- First of all, it can make *typing derivations very large* (if a location *appears many times* in a term) !
- e.g., if

$$
\begin{aligned}
\mu = \; & l_1 \mapsto \lambda x\!: \mathrm{Nat}.\ 999, \\
& l_2 \mapsto \lambda x\!: \mathrm{Nat}.\ (!\,l_1)\ x, \\
& l_3 \mapsto \lambda x\!: \mathrm{Nat}.\ (!\,l_2)\ x, \\
& l_4 \mapsto \lambda x\!: \mathrm{Nat}.\ (!\,l_3)\ x, \\
& l_5 \mapsto \lambda x\!: \mathrm{Nat}.\ (!\,l_4)\ x\,),
\end{aligned}
$$

then how big is the typing derivation for $!\,l_5$?

Problems #2

But wait... it *gets worse* if the store contains a *cycle*. Suppose

$$\mu = l_1 \mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_2)\ x,$$
$$\quad l_2 \mapsto \lambda x{:}\,\mathrm{Nat}.\ (!\,l_1)\ x),$$

how big is the typing derivation for $!\,l_2$?

Calculating a type for l_2 requires finding the type of l_1, which in turn involves l_2.

Why?

What leads to the problems?

Our typing rule for locations requires us to *recalculate the type of a location every time it's* mentioned in a term, which should not be necessary.

In fact, once a location is first created, *the type of the initial value* is known, and *the type will be kept* even if the values can be changed.

Store Typing

Observation:

The typing rules we have chosen for references guarantee *that a given location* in the store is *always* used to hold *values of the same type*.

These intended types can be collected into a ***store typing:***

— a *partial function* from *locations* to *types*.

Store Typing

E.g., for

$$\begin{aligned}\mu = (\, & l_1 \mapsto \lambda x{:}\,\text{Nat}.\ 999, \\ & l_2 \mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_1)\ x, \\ & l_3 \mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_2)\ x, \\ & l_4 \mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_3)\ x, \\ & l_5 \mapsto \lambda x{:}\,\text{Nat}.\ (!\,l_4)\ x\,),\end{aligned}$$

A reasonable *store typing* would be

$$\begin{aligned}\Sigma = (\, & l_1 \mapsto \text{Nat}\to\text{Nat}, \\ & l_2 \mapsto \text{Nat}\to\text{Nat}, \\ & l_3 \mapsto \text{Nat}\to\text{Nat}, \\ & l_4 \mapsto \text{Nat}\to\text{Nat}, \\ & l_5 \mapsto \text{Nat}\to\text{Nat}\,)\end{aligned}$$

Store Typing

Now, suppose we are given *a store typing* Σ describing the store μ in which we intend to evaluate some term t. Then we can use Σ to look up the types of locations in t instead of calculating them from the values in μ.

$$\frac{\Sigma(l) = \mathrm{T}_1}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref}\ \mathrm{T}_1} \qquad \text{(T-Loc)}$$

i.e., *typing* is now a *four-place relation on* contexts, *store typings*, terms, and types.

Proviso: the typing rules accurately predict the results of evaluation *only if* the concrete store used during evaluation actually *conforms to* the store typing.

Final typing rules

$$\frac{\Sigma(l) = \mathtt{T}_1}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Loc)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \mid \Sigma \vdash \mathtt{ref}\ \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Ref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash \mathtt{!t}_1 : \mathtt{T}_{11}} \quad \text{(T-Deref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11} \qquad \Gamma \mid \Sigma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash \mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 : \mathtt{Unit}} \quad \text{(T-Assign)}$$

Store Typing

Question: Where do *these store typings* come from?

Answer: When we first typecheck a program, there will be no explicit locations, so we can use *an empty store typing*, since the locations arise only in terms that are *the intermediate results* of evaluation.

So, when a new location is created during evaluation,

$$\frac{l \notin dom(\mu)}{\texttt{ref}\ \ \texttt{v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \texttt{v}_1)} \qquad \text{(E-RefV)}$$

we can observe the type of $\texttt{v}_1$ and extend the "current store typing" appropriately.

Store Typing

As evaluation proceeds and new locations are created, *the store typing is extended* by looking at the type of the initial values being placed in newly allocated cells.

Σ only records the *association* between *already-allocated storage cells* and *their types*.

Safety

Preservation

Preservation

How to express the statement of preservation?

First attempt: just add *stores* and *store typings* in the appropriate places.

Theorem(?): if $\Gamma \mid \Sigma \vdash t : T$ and $t|\mu \longrightarrow t'|\mu'$, then $\Gamma \mid \Sigma \vdash t' : T$

Right??

Wrong!

Why wrong?

Because Σ and μ here are not constrained to have anything to do with each other!

Preservation

Definition: A store μ is said to be *well typed* with respect to a typing context Γ and a store typing Σ, written $\Gamma \mid \Sigma \vdash \mu$, if $dom(\mu) = dom(\Sigma)$ and $\Gamma \mid \Sigma \vdash \mu(l){:}\ \Sigma(l)$ for every $l \in dom(\mu)$.

Theorem (?) : if

$$\Gamma \mid \Sigma \vdash t : T$$
$$t \mid \mu \longrightarrow t' \mid \mu'$$
$$\Gamma \mid \Sigma \vdash \mu$$

then $\Gamma \mid \Sigma \vdash t' : T$

Right this time?

Still wrong !

Why？

Preservation

Creation of a *new reference cell* ...

$$\frac{l \notin dom(\mu)}{\mathrm{ref}\ \mathrm{v}_1 \,|\, \mu \longrightarrow l \,|\, (\mu, l \mapsto \mathrm{v}_1)} \qquad \text{(E-RefV)}$$

... *breaks the correspondence* between the store typing and the store.

Since *the store can grow during evaluation*:

Creation of a new reference cell yields a store with a *larger domain* than the initial one, making the conclusion incorrect: if μ' includes a binding for a fresh location l, then l cann't be in the domain of Σ, and it will not be the case that t' is typable under Σ.

Preservation

Theorem: if

$$\Gamma \mid \Sigma \vdash \mathrm{t} : \mathrm{T}$$
$$\Gamma \mid \Sigma \vdash \mu$$
$$\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$$

then, for *some* $\Sigma' \supseteq \Sigma$,

$$\Gamma \mid \Sigma' \vdash \mathrm{t}' : \mathrm{T}$$
$$\Gamma \mid \Sigma' \vdash \mu'.$$

A correct version.

What is Σ' ?

Proof: Easy extension of the preservation proof for $\lambda_{\rightarrow}$.

Progress

Progress

Theorem:

Suppose t is a closed, well-typed term (*that is*, $\Gamma | \Sigma \vdash \mathrm{t} : \mathrm{T}$ for some T and Σ).

Then either t is a *value* or else, for any store μ such that $\Gamma | \Sigma \vdash \mu$, there is some term t' and store μ' with $\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$.

In summary …

Syntax

We added to $\lambda_{\rightarrow}$ (with Unit) syntactic forms for *creating*, *dereferencing*, and *assigning* reference cells, plus a new type constructor Ref.

t ::=		*terms*
	unit	*unit constant*
	x	*variable*
	λx:T.t	*abstraction*
	t t	*application*
	ref t	*reference creation*
	!t	*dereference*
	t:=t	*assignment*
	l	store location

Evaluation

Evaluation becomes *a four-place* relation: $t \mid \mu \longrightarrow t' \mid \mu'$

$$\frac{l \notin dom(\mu)}{\texttt{ref } \mathtt{v}_1 \mid \mu \longrightarrow l \mid (\mu, l \mapsto \mathtt{v}_1)} \qquad \text{(E-RefV)}$$

$$\frac{\mu(l) = \mathtt{v}}{!l \mid \mu \longrightarrow \mathtt{v} \mid \mu} \qquad \text{(E-DerefLoc)}$$

$$l := \mathtt{v}_2 \mid \mu \longrightarrow \texttt{unit} \mid [l \mapsto \mathtt{v}_2]\mu \qquad \text{(E-Assign)}$$

Typing

Typing becomes *a three-place* relation: $\Gamma \mid \Sigma \vdash \mathtt{t} : \mathtt{T}$

$$\frac{\Sigma(l) = \mathtt{T}_1}{\Gamma \mid \Sigma \vdash l : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Loc)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{T}_1}{\Gamma \mid \Sigma \vdash \mathtt{ref}\ \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_1} \quad \text{(T-Ref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash\ !\mathtt{t}_1 : \mathtt{T}_{11}} \quad \text{(T-Deref)}$$

$$\frac{\Gamma \mid \Sigma \vdash \mathtt{t}_1 : \mathtt{Ref}\ \mathtt{T}_{11} \qquad \Gamma \mid \Sigma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \mid \Sigma \vdash \mathtt{t}_1 \mathtt{:=} \mathtt{t}_2 : \mathtt{Unit}} \quad \text{(T-Assign)}$$

Preservation

Theorem: if

$$\Gamma \mid \Sigma \vdash t : T$$

$$\Gamma \mid \Sigma \vdash \mu$$

$$t \mid \mu \longrightarrow t' \mid \mu'$$

then, for some $\Sigma' \supseteq \Sigma$,

$$\Gamma \mid \Sigma' \vdash t' : T$$

$$\Gamma \mid \Sigma' \vdash \mu'.$$

Progress

Theorem: Suppose t is a *closed, well-typed* term (that is, $\emptyset \mid \Sigma \vdash \mathrm{t} : \mathrm{T}$ for some T and Σ). Then either t is a value or else, for any store μ such that $\emptyset \mid \Sigma \vdash \mu$, there is some term t' and store μ' with $\mathrm{t} \mid \mu \longrightarrow \mathrm{t}' \mid \mu'$.

Others …

Arrays

Fix-sized vectors of values. All of the values must have the *same type*, and the fields in the array can be accessed and modified.

e.g., in Ocaml, arrays can be created with

[|e_1; … ; e_n|]

let a = [|1;3;5;7;9|];;

val a : int array = [|1;3;5;7;9|]

#a;;

-: int array = [|1;3;5;7;9|]

Recursion via references

Indeed, we can define *arbitrary recursive functions* using references.

1. Allocate a ref cell and initialize it with a *dummy function* of the appropriate type:

$$\text{fact}_{ref} \;=\; \text{ref}\,(\lambda n{:}\,\text{Nat}.\,0)$$

2. Define the body of the function we are interested in, using the contents of the reference cell for making recursive calls:

$$\text{fact}_{body} =$$
$$\lambda n{:}\,\text{Nat}.$$
$$\text{if iszero n then 1 else times n }((!\,\text{fact}_{ref})(\text{pred n}))$$

3. "Backpatch" by storing the real body into the reference cell:

$$\text{fact}_{ref} := \text{fact}_{body}$$

4. Extract the contents of the reference cell and use it as desired:

$$\text{fact} \;=\; !\,\text{fact}_{ref}$$
$$\text{fact } 5$$

Homework☺

- Read chapter 13
- Read and chew over the codes of *fullref*.

- HW: 13.1.2 and 13.5.8

- Preview chapter 14

Non-termination via references

There are *well-typed terms* in this system that are not strongly normalizing. For example:

$$t1 = \lambda r{:}\ \mathrm{Ref}\ (\mathrm{Unit} \to \mathrm{Unit}).$$

$$(r \ := (\lambda x{:}\ \mathrm{Unit}.\ (!\ r)x);$$

$$(!\ r)\ \mathrm{unit});$$

$$t2 = \mathrm{ref}\ (\lambda x{:}\ \mathrm{Unit}.\ x);$$

Applying t1 to t2 yields a (well-typed) divergent term.

Nontermination via references

There are well-typed terms in this system that are not strongly normalizing. For example:

$$
\begin{aligned}
t1 &= \lambda r\!: Ref\,(Unit \to Unit).\\
&\quad \boxed{(r \;:= (\lambda x\!: Unit.\,(!\,r)x);}\\
&\quad (!\,r)\ unit);\\
t2 &= ref\,(\lambda x\!: Unit.\,x);
\end{aligned}
$$

Applying t1 to t2 yields a (well-typed) divergent term.