Chap 16

Metatheory of Subtyping

Algorithmic Subtyping

Algorithmic Typing

Joins and Meets

Developing an algorithmic subtyping relation

Subtype Relation

$$\mathtt{S} <: \mathtt{S} \qquad \text{(S-Refl)}$$

$$\frac{\mathtt{S} <: \mathtt{U} \qquad \mathtt{U} <: \mathtt{T}}{\mathtt{S} <: \mathtt{T}} \qquad \text{(S-Trans)}$$

$$\{\mathtt{l}_i\mathtt{:T}_i{}^{i\in 1..n+k}\} <: \{\mathtt{l}_i\mathtt{:T}_i{}^{i\in 1..n}\} \qquad \text{(S-RcdWidth)}$$

$$\frac{\text{for each } i \qquad \mathtt{S}_i <: \mathtt{T}_i}{\{\mathtt{l}_i\mathtt{:S}_i{}^{i\in 1..n}\} <: \{\mathtt{l}_i\mathtt{:T}_i{}^{i\in 1..n}\}} \qquad \text{(S-RcdDepth)}$$

$$\frac{\{\mathtt{k}_j\mathtt{:S}_j{}^{j\in 1..n}\} \text{ is a permutation of } \{\mathtt{l}_i\mathtt{:T}_i{}^{i\in 1..n}\}}{\{\mathtt{k}_j\mathtt{:S}_j{}^{j\in 1..n}\} <: \{\mathtt{l}_i\mathtt{:T}_i{}^{i\in 1..n}\}} \qquad \text{(S-RcdPerm)}$$

$$\frac{\mathtt{T}_1 <: \mathtt{S}_1 \qquad \mathtt{S}_2 <: \mathtt{T}_2}{\mathtt{S}_1 \rightarrow \mathtt{S}_2 <: \mathtt{T}_1 \rightarrow \mathtt{T}_2} \qquad \text{(S-Arrow)}$$

$$\mathtt{S} <: \mathtt{Top} \qquad \text{(S-Top)}$$

Issues in Subtyping

For a *given subtyping statement*, there are *multiple rules* that could be used in a derivation.

1. The conclusions of S-RcdWidth, S-RcdDepth, and S-RcdPerm *overlap with each other*.
2. S-REFL and S-TRANS overlap with every other rule.

What to do?

We'll turn the *declarative version* of subtyping into the *algorithmic version*.

The problem was that we don't have an algorithm to decide when $S <: T$ or $\Gamma \vdash t : T$.

Both sets of rules are not *syntax-directed.*

Syntax-directed rules

In the simply typed lambda-calculus (without subtyping), each rule can be "*read from bottom to top*" in a straightforward way.

$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \qquad \text{(T-APP)}$$

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If we are given some Γ and some t of the form $t_1\ \ t_2$, we can try to *find a type* for t by

1. finding (recursively) a type for t_1
2. checking that it has the form $T_{11} \longrightarrow T_{12}$
3. finding (recursively) a type for t_2
4. checking that it is the same as T_{11}

Syntax-directed rules

Technically, the reason this works is that we can *divide the "**positions**"* of the typing relation into ***input positions*** (i.e., Γ and $\mathtt{t}$) and ***output positions*** ($\mathtt{T}$).

- For the input positions, all metavariables appearing in the *premises* also appear in the *conclusion* (so we can calculate inputs to the *"subgoals"* from the subexpressions of inputs to the main goal)
- For the output positions, all metavariables appearing in the *conclusions* also appear in the *premises* (so we can calculate outputs from the main goal from the outputs of the subgoals)

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \to \mathtt{T}_{12} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{T}_{12}} \qquad \text{(T-APP)}$$

Syntax-directed sets of rules

The *second important point* about the simply typed lambda-calculus is that *the set of typing rules is syntax-directed*, in the sense that, for every "*input*" Γ and t, *there is one rule* that can be used to derive typing statements involving t.

E.g., if t is an *application*, then we must proceed by trying to use T-App. If we succeed, then we have found a type (indeed, the *unique type*) for t. If it *fails,* then we know that t is *not typable*.

$\Longrightarrow$ no backtracking!

Non-syntax-directedness of typing

When we extend the system with *subtyping*, both aspects of syntax-directedness get broken.

1. The set of typing rules now includes *two* rules that can be used to give a type to terms of a given shape (the old one plus T–SUB)

$$\frac{\Gamma \vdash t : S \qquad S <: T}{\Gamma \vdash t : T} \qquad \text{(T-SUB)}$$

2. Worse yet, the new rule T-SUB itself is not syntax directed: the *inputs* to the left-hand subgoal are exactly the same as the *inputs* to the main goal!

   (Hence, if we translated the typing rules naively into a typechecking function, the case corresponding to T-SUB would cause divergence.)

Non-syntax-directedness of subtyping

Moreover, the *subtyping relation* is *not syntax directed* either.

1. There are *lots* of ways to derive a given subtyping statement.
2. The transitivity rule

$$\frac{\mathtt{S} <: \mathtt{U} \qquad \mathtt{U} <: \mathtt{T}}{\mathtt{S} <: \mathtt{T}} \qquad \text{(S-TRANS)}$$

is *badly non-syntax-directed*: the premises contain a *metavariable* (in an "*input position*") that does *not appear at all in the conclusion*.

To implement this rule naively, we have to *guess* a value for U!

What to do?

1. *Observation*: We don't *need* lots of ways to prove a given typing or subtyping statement — *one is enough*.

   $\longrightarrow$ *Think more carefully about the typing and subtyping systems to see where we can get rid of excess flexibility.*

2. Use the resulting intuitions to formulate new "*algorithmic*" (i.e., syntax-directed) typing and subtyping relations.
3. Prove that the algorithmic relations are "*the same as*" the original ones in an appropriate sense.

Algorithmic Subtyping

What to do

How do we change the rules deriving $S <: T$ to be *syntax-directed*?

There are lots of ways to derive a given subtyping statement $S <: T$.

The general idea is to *change this system* so that there is *only one way* to derive it.

Step 1: simplify record subtyping

Idea: combine all three record subtyping rules into one "*macro rule*" that captures all of their effects

$$\frac{\{\mathtt{l}_i{}^{i\in 1..n}\} \subseteq \{\mathtt{k}_j{}^{j\in 1..m}\} \qquad \mathtt{k}_j = \mathtt{l}_i \text{ implies } \mathtt{S}_j <: \mathtt{T}_i}{\{\mathtt{k}_j\mathtt{:}\mathtt{S}_j{}^{j\in 1..m}\} <: \{\mathtt{l}_i\mathtt{:}\mathtt{T}_i{}^{i\in 1..n}\}} \qquad \text{(S-RCD)}$$

Simpler subtype relation

$$\mathtt{S} <: \mathtt{S} \qquad \text{(S-Refl)}$$

$$\frac{\mathtt{S} <: \mathtt{U} \qquad \mathtt{U} <: \mathtt{T}}{\mathtt{S} <: \mathtt{T}} \qquad \text{(S-Trans)}$$

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$$\frac{\mathtt{T}_1 <: \mathtt{S}_1 \qquad \mathtt{S}_2 <: \mathtt{T}_2}{\mathtt{S}_1 \to \mathtt{S}_2 <: \mathtt{T}_1 \to \mathtt{T}_2} \qquad \text{(S-Arrow)}$$

$$\mathtt{S} <: \mathtt{Top} \qquad \text{(S-Top)}$$

Step 2: Get rid of reflexivity

Observation: S-REFL is unnecessary.

Lemma: $S <: S$ can be derived for every type S without using S-REFL.

Even simpler subtype relation

$$\frac{\mathtt{S} <: \mathtt{U} \qquad \mathtt{U} <: \mathtt{T}}{\mathtt{S} <: \mathtt{T}} \qquad \text{(S-Trans)}$$

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$$\mathtt{S} <: \mathtt{Top} \qquad \text{(S-Top)}$$

Step 3: Get rid of transitivity

Observation: S-Trans is unnecessary.

Lemma: If $S <: T$ can be derived, then it can be derived without using S-Trans .

Even simpler subtype relation

$$\frac{\{\mathtt{l}_i{}^{\,i\in 1..n}\} \subseteq \{\mathtt{k}_j{}^{\,j\in 1..m}\} \qquad \mathtt{k}_j = \mathtt{l}_i \text{ implies } \mathtt{S}_j <: \mathtt{T}_i}{\{\mathtt{k}_j\!:\!\mathtt{S}_j{}^{\,j\in 1..m}\} <: \{\mathtt{l}_i\!:\!\mathtt{T}_i{}^{\,i\in 1..n}\}} \qquad \text{(S-RCD)}$$

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$$\mathtt{S} <: \mathtt{Top} \qquad \text{(S-TOP)}$$

"Algorithmic" subtype relation

$$\boxed{\;\mapsto} S <: \mathtt{Top} \qquad \text{(}\boxed{\text{SA}}\text{-Top)}$$

$$\frac{\mapsto T_1 <: S_1 \qquad \mapsto S_2 <: T_2}{\mapsto S_1 \rightarrow S_2 <: T_1 \rightarrow T_2} \qquad \text{(SA-Arrow)}$$

$$\frac{\{\mathtt{l}_i{}^{i\in 1..n}\} \subseteq \{\mathtt{k}_j{}^{j\in 1..m}\} \qquad \text{for each } \mathtt{k}_j = \mathtt{l}_i,\ \mapsto S_j <: T_i}{\mapsto \{\mathtt{k}_j{:}S_j{}^{j\in 1..m}\} <: \{\mathtt{l}_i{:}T_i{}^{i\in 1..n}\}} \qquad \text{(SA-Rcd)}$$

Soundness and completeness

Theorem: $S <: T$ iff $\mapsto S <: T$

Terminology:

- The algorithmic presentation of subtyping is *sound* with respect to the original, if $\mapsto S <: T$ implies $S <: T$. (Everything validated by the algorithm is actually true.)
- The algorithmic presentation of subtyping is *complete* with respect to the original, if $S <: T$ implies $\mapsto S <: T$. (Everything true is validated by the algorithm.)

Decision Procedures

A *decision procedure* for a relation $R \subseteq U$ is *a total function* p from U to *{true, false}* such that $p(u) = true$ iff $u \in R$.

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Is our *subtype* function a decision procedure?

Since *subtype* is just an implementation of the algorithmic subtyping rules, we have

1. if $subtype(\mathrm{S}, \mathrm{T}) = true$, then $\mapsto \mathrm{S} <: \mathrm{T}$ (hence, by soundness of the algorithmic rules, $\mathrm{S} <: \mathrm{T}$)
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Q: What's missing?

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Q: What's missing?

A: How do we know that *subtype* is a *total function*?

Prove it!

Decision Procedures

Recall: A *decision procedure* for a relation $R \subseteq U$ is *a total function* p from U to *{true, false}* such that $p(u) = true$ iff $u \in R$.

Example:

$$U = \{1, 2, 3\}$$
$$R = \{(1, 2), (2, 3)\}$$

Note that, we are saying nothing about *computability.*

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The function p whose graph is

{ ((1, 2), *true*), ((2, 3), *true*),
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is a decision function for R.

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The function p'' whose graph is

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is also *not* a decision function for R.

Decision Procedures (take 2)

We want a decision procedure to be a *procedure*.

A *decision procedure* for a relation $R \subseteq U$ is a *computable total function* p from U to *{true, false}* such that $p(u) = true$ iff $\mathrm{u} \in R$.

Example

$$U = \{1, 2, 3\}$$
$$R = \{(1, 2), (2, 3)\}$$

The function

$$\begin{aligned} p(x, y) = \ & if\ x = 2\ and\ y = 3\ then\ true \\ & else\ if\ x = 1\ and\ y = 2\ then\ true \\ & else\ false \end{aligned}$$

whose graph is

{ ((1, 2), *true*), ((2, 3), *true*),
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is a decision procedure for R.

Example

$$U = \{1,2,3\}$$

$$R = \{(1,2),(2,3)\}$$

The recursively defined *partial function*

$$p(x,y) = if\ x = 2\ and\ y = 3\ then\ true$$
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whose graph is

{ ((1, 2), *true*), ((2, 3), *true*), ((1, 3), *false*)}

is *not* a decision procedure for R.

Subtyping Algorithm

This *recursively defined total function* is a decision procedure for the subtype relation:

subtype(S, T) =

if $\mathrm{T} = \mathrm{Top}$, then *true*

else if $\mathrm{S} = \mathrm{S}_1 \longrightarrow \mathrm{S}_2$ and $\mathrm{T} = \mathrm{T}_1 \longrightarrow \mathrm{T}_2$

then $subtype(\mathrm{T}_1, \mathrm{S}_1) \wedge subtype(\mathrm{S}_2, \mathrm{T}_2)$

else if $\mathrm{S} = \{\mathrm{k}_\mathrm{j}:\ \mathrm{S}_\mathrm{j}{}^{\mathrm{j}\in 1..\mathrm{m}}\}$ and $\mathrm{T} = \{\mathrm{l}_\mathrm{i}:\ \mathrm{T}_\mathrm{i}{}^{\mathrm{i}\in 1..n}\}$

then $\{\mathrm{l}_\mathrm{i}{}^{\mathrm{i}\in 1..\mathrm{n}}\} \subseteq \{\mathrm{k}_\mathrm{j}{}^{\mathrm{j}\in 1..\mathrm{m}}\}$

$\wedge$ for all $i \in 1..n$ there is some $j \in 1..m$ with $\mathrm{k}_\mathrm{j} = \mathrm{l}_\mathrm{i}$ and $subtype(\mathrm{S}_\mathrm{j}, \mathrm{T}_\mathrm{i})$

else *false*.

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and $subtype(\mathrm{S}_\mathrm{j}, \mathrm{T}_\mathrm{i})$

else *false*.

To show this, we need to prove:

1. that it returns *true* whenever $\mathrm{S} <: \mathrm{T}$, and
2. that it returns either *true* or *false* on all inputs.

Algorithmic Typing

Algorithmic typing

How do we implement a *type checker* for the lambda-calculus *with subtyping*?

Given a context Γ and a term t, how do we determine its type T, such that $\Gamma \vdash t : T$?

Issue

For the typing relation, we have *just one problematic rule* to deal with: subsumption rule

$$\frac{\Gamma \vdash \mathtt{t} : \mathtt{S} \qquad \mathtt{S} <: \mathtt{T}}{\Gamma \vdash \mathtt{t} : \mathtt{T}} \qquad \text{(T-SUB)}$$

Q: where is this rule really needed?

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Q: where is this rule really needed?

For applications, e.g., the term

$$(\lambda r{:}\{x{:}\,Nat\}.\ r.x)\ \{x = 0, y = 1\}$$

is *not typable* without using subsumption.

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Where else??

Nowhere else!

Uses of subsumption to help typecheck *applications* are the only interesting ones.

Plan

1. Investigate *how subsumption is used* in typing derivations by *looking at examples* of how it can be "pushed through" other rules
2. Use the intuitions gained from these examples to design a new, algorithmic typing relation that
   - *Omits subsumption*
   - Compensates for its absence by *enriching the application rule*
3. *Show that* the algorithmic typing relation is essentially equivalent to the original, declarative one

Example (T-ABS)

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, x{:}S_1 \vdash s_2 : S_2} \qquad \dfrac{\vdots}{S_2 <: T_2}}{\Gamma, x{:}S_1 \vdash s_2 : T_2}\ (\text{T-SUB})}{\Gamma \vdash \lambda x{:}S_1.s_2 : S_1 \to T_2}\ (\text{T-ABS})
$$

Example (T-Abs)

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, x{:}S_1 \vdash s_2 : S_2} \qquad \dfrac{\vdots}{S_2 <: T_2}}{\Gamma, x{:}S_1 \vdash s_2 : T_2}\;\text{(T-Sub)}}{\Gamma \vdash \lambda x{:}S_1.s_2 : S_1 \to T_2}\;\text{(T-Abs)}$$

becomes

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, x{:}S_1 \vdash s_2 : S_2}}{\Gamma \vdash \lambda x{:}S_1.s_2 : S_1 \to S_2}\;\text{(T-Abs)} \qquad \dfrac{\dfrac{}{S_1 <: S_1}\;\text{(S-Refl)} \qquad \dfrac{\vdots}{S_2 <: T_2}}{S_1 \to S_2 <: S_1 \to T_2}\;\text{(S-Arrow)}}{\Gamma \vdash \lambda x{:}S_1.s_2 : S_1 \to T_2}\;\text{(T-Sub)}$$

Intuitions

These examples show that we do not need T-SUB to "enable" T-ABS : given any typing derivation, we can construct a derivation *with the same conclusion* in which T-SUB is never used immediately before T-ABS.

What about T-APP?

We've already observed that T-SUB is required for typechecking some *applications*.

So we expect to find that we ***cannot*** play the same game with T-APP as we've done with T-ABS.

Let's see why.

Example (T−Sub with T-APP on the left)

$$
\dfrac{
  \dfrac{
    \dfrac{\vdots}{\Gamma \vdash s_1 : S_{11} \to S_{12}}
    \qquad
    \dfrac{
      \dfrac{\vdots}{T_{11} <: S_{11}}
      \qquad
      \dfrac{\vdots}{S_{12} <: T_{12}}
    }{S_{11} \to S_{12} <: T_{11} \to T_{12}}\;(\text{S-ARROW})
  }{\Gamma \vdash s_1 : T_{11} \to T_{12}}\;(\text{T-SUB})
  \qquad
  \dfrac{\vdots}{\Gamma \vdash s_2 : T_{11}}
}{\Gamma \vdash s_1\ s_2 : T_{12}}\;(\text{T-APP})
$$

becomes

$$
\dfrac{
  \dfrac{
    \dfrac{\vdots}{\Gamma \vdash s_1 : S_{11} \to S_{12}}
    \qquad
    \dfrac{
      \dfrac{\vdots}{\Gamma \vdash s_2 : T_{11}}
      \qquad
      \dfrac{\vdots}{T_{11} <: S_{11}}
    }{\Gamma \vdash s_2 : S_{11}}\;(\text{T-SUB})
  }{\Gamma \vdash s_1\ s_2 : S_{12}}\;(\text{T-APP})
  \qquad
  \dfrac{\vdots}{S_{12} <: T_{12}}
}{\Gamma \vdash s_1\ s_2 : T_{12}}\;(\text{T-SUB})
$$

Example (T—Sub with T-APP on the right)

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash s_1 : T_{11}\to T_{12}} \qquad \dfrac{\dfrac{\vdots}{\Gamma \vdash s_2 : T_2} \qquad \dfrac{\vdots}{T_2 <: T_{11}}}{\Gamma \vdash s_2 : T_{11}}\,(\text{T-Sub})}{\Gamma \vdash s_1\ s_2 : T_{12}}\,(\text{T-App})$$

becomes

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash s_1 : T_{11}\to T_{12}} \qquad \dfrac{\dfrac{\vdots}{T_2 <: T_{11}} \qquad \dfrac{}{T_{12} <: T_{12}}\,(\text{S-Refl})}{T_{11}\to T_{12} <: T_2\to T_{12}}\,(\text{S-Arrow})}{\Gamma \vdash s_1 : T_2\to T_{12}}\,(\text{T-Sub}) \qquad \dfrac{\vdots}{\Gamma \vdash s_2 : T_2}}{\Gamma \vdash s_1\ s_2 : T_{12}}\,(\text{T-App})$$

Observations

So we've seen that uses of subsumption can be "*pushed*" from one of immediately before T-APP's premises to the other, but *cannot be completely eliminated*.

Example (nested uses of T-Sub)

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash s : S} \qquad \dfrac{\vdots}{S <: U}}{\Gamma \vdash s : U}\;(\text{T-Sub}) \qquad \dfrac{\vdots}{U <: T}}{\Gamma \vdash s : T}\;(\text{T-Sub})$$

Example (nested uses of T-Sub)

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash s : S} \qquad \dfrac{\vdots}{S <: U}}{\Gamma \vdash s : U}\,\text{(T-Sub)} \qquad \dfrac{\vdots}{U <: T}}{\Gamma \vdash s : T}\,\text{(T-Sub)}$$

becomes

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash s : S} \qquad \dfrac{\dfrac{\vdots}{S <: U} \qquad \dfrac{\vdots}{U <: T}}{S <: T}\,\text{(S-Trans)}}{\Gamma \vdash s : T}\,\text{(T-Sub)}$$

Summary

What we've learned:

- Uses of the T-Sub rule can be "*pushed down*" through typing derivations until they encounter either
  1. a use of T-App or
  2. the root of the derivation tree.
- In both cases, multiple uses of T-Sub can be coalesced into a single one.

Summary

What we've learned:

- Uses of the T-Sub rule can be "pushed down" through typing derivations until they encounter either
  1. a use of T-App or
  2. the root of the derivation tree.
- In both cases, multiple uses of T-Sub can be collapsed into a single one.

This suggests a notion of "normal form" for typing derivations, in which there is

- exactly one use of T-Sub before each use of T-App
- one use of T-Sub at the very end of the derivation
- no uses of T T-Sub anywhere else.

Algorithmic Typing

The next step is to "build in" the use of subsumption in application rules, by changing the T-App rule to incorporate a subtyping premise.

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_2 \qquad \vdash T_2 <: T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}$$

Given any typing derivation, we can now

1. normalize it, to move all uses of subsumption to either just before applications (in the right-hand premise) or at the very end
2. replace uses of T-App with T-SUB in the right-hand premise by uses of the extended rule above

This yields a derivation in which there is just *one* use of subsumption, at the very end!

Minimal Types

But... if subsumption is only used at the very end of derivations, then it is actually *not needed* in order to show that *any term is typable*!

It is just used to give *more* types to terms that have already been shown to have a type.

In other words, if we dropped subsumption completely (after refining the application rule), we would still be able to give types to exactly the same set of terms — we just would not be able to give as many types to some of them.

If we drop subsumption, then the remaining rules will assign a *unique*, *minimal* type to each typable term.

For purposes of building a typechecking algorithm, this is enough.

Final Algorithmic Typing Rules

$$\frac{\mathtt{x}{:}\mathtt{T} \in \Gamma}{\Gamma \blacktriangleright \mathtt{x} : \mathtt{T}} \qquad \text{(TA-VAR)}$$

$$\frac{\Gamma, \mathtt{x}{:}\mathtt{T}_1 \blacktriangleright \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \blacktriangleright \lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_2 : \mathtt{T}_1 \to \mathtt{T}_2} \qquad \text{(TA-ABS)}$$

$$\frac{\Gamma \blacktriangleright \mathtt{t}_1 : \mathtt{T}_1 \qquad \mathtt{T}_1 = \mathtt{T}_{11} \to \mathtt{T}_{12} \qquad \Gamma \blacktriangleright \mathtt{t}_2 : \mathtt{T}_2 \qquad \blacktriangleright \mathtt{T}_2 <: \mathtt{T}_{11}}{\Gamma \blacktriangleright \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{T}_{12}} \qquad \text{(TA-APP)}$$

$$\frac{\text{for each } i \qquad \Gamma \blacktriangleright \mathtt{t}_i : \mathtt{T}_i}{\Gamma \blacktriangleright \{\mathtt{l}_1{=}\mathtt{t}_1 \ldots \mathtt{l}_n{=}\mathtt{t}_n\} : \{\mathtt{l}_1{:}\mathtt{T}_1 \ldots \mathtt{l}_n{:}\mathtt{T}_n\}} \qquad \text{(TA-RCD)}$$

$$\frac{\Gamma \blacktriangleright \mathtt{t}_1 : \mathtt{R}_1 \qquad \mathtt{R}_1 = \{\mathtt{l}_1{:}\mathtt{T}_1 \ldots \mathtt{l}_n{:}\mathtt{T}_n\}}{\Gamma \blacktriangleright \mathtt{t}_1.\mathtt{l}_i : \mathtt{T}_i} \qquad \text{(TA-PROJ)}$$

Completeness of the algorithmic rules

Theorem [Minimal Typing]: If $\Gamma \vdash t : T$, then $\Gamma \mapsto t : S$ for some $S <: T$.

Completeness of the algorithmic rules

Theorem [Minimal Typing]: If $\Gamma \vdash t : T$, then $\Gamma \mapsto t : S$ for some $S <: T$.

Proof: Induction on *typing derivation*.

(N.b.: All the messing around with transforming derivations was just to build intuitions and *decide what algorithmic rules* to write down and *what property* to prove: the proof itself is a straightforward induction on typing derivations.)

Meets and Joins

Adding Booleans

Suppose we want to add *booleans* and *conditionals* to the language we have been discussing.

For the declarative presentation of the system, we just add in the appropriate syntactic forms, evaluation rules, and typing rules.

$$\Gamma \vdash \mathtt{true} : \mathtt{Bool} \qquad \text{(T-TRUE)}$$

$$\Gamma \vdash \mathtt{false} : \mathtt{Bool} \qquad \text{(T-FALSE)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Bool} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T} \qquad \Gamma \vdash \mathtt{t}_3 : \mathtt{T}}{\Gamma \vdash \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 : \mathtt{T}} \qquad \text{(T-IF)}$$

A Problem with Conditional Expressions

For the algorithmic presentation of the system, however, we encounter a little difficulty.

What is the minimal type of

$if\ true\ then\ \{x = true, y = false\}\ else\ \{x = true, z = ture\}$?

The Algorithmic Conditional Rule

More generally, we can use subsumption to give an expression

$$\text{if } t_1 \text{ then } t_2 \text{ else } t_3$$

any type that is a possible type of both t_2 and t_3.

So the *minimal* type of the conditional is the *least common supertype* (or *join*) of the minimal type of t_2 and the minimal type of t_3.

$$\frac{\Gamma \mathrel{\blacktriangleright} t_1 : \text{Bool} \qquad \Gamma \mathrel{\blacktriangleright} t_2 : T_2 \qquad \Gamma \mathrel{\blacktriangleright} t_3 : T_3}{\Gamma \mathrel{\blacktriangleright} \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T_2 \vee T_3} \quad \text{(T-IF)}$$

The Algorithmic Conditional Rule

More generally, we can use subsumption to give an expression

$$\text{if } t_1 \text{ then } t_2 \text{ else } t_3$$

any type that is a possible type of both t_2 and t_3.

So the *minimal* type of the conditional is the *least common supertype* (or *join*) of the minimal type of t_2 and the minimal type of t_3.

$$\frac{\Gamma \blacktriangleright t_1 : \texttt{Bool} \qquad \Gamma \blacktriangleright t_2 : T_2 \qquad \Gamma \blacktriangleright t_3 : T_3}{\Gamma \blacktriangleright \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T_2 \vee T_3} \quad \text{(T-If)}$$

Q: Does such a type exist for every T_2 and T_3??

Existence of Joins

Theorem: For every pair of types S and T, there is a type J such that

1. $S <: J$
2. $T <: J$
3. If K is a type such that $S <: K$ and $T <: K$, then $J <: K$.

i.e., J is the smallest type that is a supertype of both S and T.

How to prove it?

Examples

What are the joins of the following pairs of types?

1. $\{x\!: \mathrm{Bool}, y\!: \mathrm{Bool}\}$ and $\{y\!: \mathrm{Bool}, z\!: \mathrm{Bool}\}$?
2. $\{x\!: \mathrm{Bool}\}$ and $\{y\!: \mathrm{Bool}\}$?
3. $\{x\!: \{a\!: \mathrm{Bool}, b\!: \mathrm{Bool}\}\}$ and $\{x\!: \{b\!: \mathrm{Bool}, c\!: \mathrm{Bool}\}, y\!: \mathrm{Bool}\}$?
4. $\{\}$ and Bool?
5. $\{x\!: \{\}\}$ and $\{x\!: \mathrm{Bool}\}$?
6. $\mathrm{Top} \longrightarrow \{x\!: \mathrm{Bool}\}$ and $\mathrm{Top} \longrightarrow \{y\!: \mathrm{Bool}\}$?
7. $\{x\!: \mathrm{Bool}\} \longrightarrow \mathrm{Top}$ and $\{y\!: \mathrm{Bool}\} \longrightarrow \mathrm{Top}$?

Meets

To calculate joins of arrow types, we also need to be able to calculate meets (greatest lower bounds)!

Unlike joins, meets do not necessarily exist.

E.g., $\mathrm{Bool} \longrightarrow \mathrm{Bool}$ and $\{\}$ have no common subtypes, so they certainly don't have a greatest one!

However...

Existence of Meets

Theorem: For every pair of types S and T, if there is any type N such that $N <: S$ and $N <: T$, then there is a type M such that

1. $M <: S$
2. $M <: T$
3. If O is a type such that $O <: S$ and $O <: T$, then $O <: M$.

i.e., M (when it exists) is the largest type that is a subtype of both S and T.

Existence of Meets

Theorem: For every pair of types S and T, if there is any type N such that $N <: S$ and $N <: T$, then there is a type M such that

1. $M <: S$
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3. If O is a type such that $O <: S$ and $O <: T$, then $O <: M$.

i.e., M (when it exists) is the largest type that is a subtype of both S and T.

Jargon: In the simply typed lambda calculus with subtyping, records, and booleans...

- The subtype relation *has joins*
- The subtype relation *has bounded meets*

Examples

What are the meets of the following pairs of types?

1. $\{x : Bool, y : Bool\}$ and $\{y : Bool, z : Bool\}$?
2. $\{x : Bool\}$ and $\{y : Bool\}$?
3. $\{x : \{a : Bool, b : Bool\}\}$ and $\{x : \{b : Bool, c : Bool\}, y : Bool\}$?
4. $\{\}$ and $Bool$?
5. $\{x : \{\}\}$ and $\{x : Bool\}$?
6. $Top \longrightarrow \{x : Bool\}$ and $Top \longrightarrow \{y : Bool\}$?
7. $\{x : Bool\} \longrightarrow Top$ and $\{y : Bool\} \longrightarrow Top$?

Calculating Joins

$$
\mathtt{S} \vee \mathtt{T} \quad = \quad \begin{cases}
\mathtt{Bool} & \text{if } \mathtt{S} = \mathtt{T} = \mathtt{Bool} \\
\mathtt{M}_1 \to \mathtt{J}_2 & \text{if } \mathtt{S} = \mathtt{S}_1 \to \mathtt{S}_2 \quad \mathtt{T} = \mathtt{T}_1 \to \mathtt{T}_2 \\
 & \quad \mathtt{S}_1 \wedge \mathtt{T}_1 = \mathtt{M}_1 \quad \mathtt{S}_2 \vee \mathtt{T}_2 = \mathtt{J}_2 \\
\{\mathtt{j}_l : \mathtt{J}_l \; ^{l \in 1..q}\} & \text{if } \mathtt{S} = \{\mathtt{k}_j : \mathtt{S}_j \; ^{j \in 1..m}\} \\
 & \quad \mathtt{T} = \{\mathtt{l}_i : \mathtt{T}_i \; ^{i \in 1..n}\} \\
 & \quad \{\mathtt{j}_l \; ^{l \in 1..q}\} = \{\mathtt{k}_j \; ^{j \in 1..m}\} \cap \{\mathtt{l}_i \; ^{i \in 1..n}\} \\
 & \quad \mathtt{S}_j \vee \mathtt{T}_i = \mathtt{J}_l \quad \text{for each } \mathtt{j}_l = \mathtt{k}_j = \mathtt{l}_i \\
\mathtt{Top} & \text{otherwise}
\end{cases}
$$

Calculating Meets

$$
S \wedge T \quad = \quad
\begin{cases}
S & \text{if } T = Top \\
T & \text{if } S = Top \\
Bool & \text{if } S = T = Bool \\
J_1 \rightarrow M_2 & \text{if } S = S_1 \rightarrow S_2 \qquad T = T_1 \rightarrow T_2 \\
 & \quad S_1 \vee T_1 = J_1 \quad S_2 \wedge T_2 = M_2 \\
\{m_l : M_l \ ^{l \in 1..q}\} & \text{if } S = \{k_j : S_j \ ^{j \in 1..m}\} \\
 & \quad T = \{l_i : T_i \ ^{i \in 1..n}\} \\
 & \quad \{m_l \ ^{l \in 1..q}\} = \{k_j \ ^{j \in 1..m}\} \cup \{l_i \ ^{i \in 1..n}\} \\
 & \quad S_j \wedge T_i = M_l \qquad \text{for each } m_l = k_j = l_i \\
 & \quad M_l = S_j \qquad\qquad \text{if } m_l = k_j \text{ occurs only in } S \\
 & \quad M_l = T_i \qquad\qquad \text{if } m_l = l_i \text{ occurs only in } T \\
fail & \text{otherwise}
\end{cases}
$$

Homework☺

- Read and digest chapter 16 & 17

- HW: 16.1.2; 16.2.6