Chapter 20: Recursive Types

Examples

Formalities

Subtyping

Review: Lists Defined in Chapter 11

- List T describes finite-length lists whose elements are drawn from T.

→ 𝔹 List *Extends λ→ (9-1) with booleans (8-1)*

New syntactic forms

t ::= ...		*terms:*
	nil[T]	*empty list*
	cons[T] t t	*list constructor*
	isnil[T] t	*test for empty list*
	head[T] t	*head of a list*
	tail[T] t	*tail of a list*
v ::= ...		*values:*
	nil[T]	*empty list*
	cons[T] v v	*list constructor*
T ::= ...		*types:*
	List T	*type of lists*

New evaluation rules $\boxed{t \longrightarrow t'}$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{cons[T]}\ t_1\ t_2 \longrightarrow \texttt{cons[T]}\ t_1'\ t_2} \quad \text{(E-Cons1)}$$

$$\frac{t_2 \longrightarrow t_2'}{\texttt{cons[T]}\ v_1\ t_2 \longrightarrow \texttt{cons[T]}\ v_1\ t_2'} \quad \text{(E-Cons2)}$$

$$\texttt{isnil[S] (nil[T])} \longrightarrow \texttt{true} \quad \text{(E-IsnilNil)}$$

$$\texttt{isnil[S] (cons[T]}\ v_1\ v_2\texttt{)} \longrightarrow \texttt{false} \quad \text{(E-IsnilCons)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{isnil[T]}\ t_1 \longrightarrow \texttt{isnil[T]}\ t_1'} \quad \text{(E-Isnil)}$$

$$\texttt{head[S] (cons[T]}\ v_1\ v_2\texttt{)} \longrightarrow v_1 \quad \text{(E-HeadCons)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{head[T]}\ t_1 \longrightarrow \texttt{head[T]}\ t_1'} \quad \text{(E-Head)}$$

$$\texttt{tail[S] (cons[T]}\ v_1\ v_2\texttt{)} \longrightarrow v_2 \quad \text{(E-TailCons)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{tail[T]}\ t_1 \longrightarrow \texttt{tail[T]}\ t_1'} \quad \text{(E-Tail)}$$

New typing rules $\boxed{\Gamma \vdash t : T}$

$$\Gamma \vdash \texttt{nil}\ [T_1] : \texttt{List}\ T_1 \quad \text{(T-Nil)}$$

$$\frac{\Gamma \vdash t_1 : T_1 \qquad \Gamma \vdash t_2 : \texttt{List}\ T_1}{\Gamma \vdash \texttt{cons}[T_1]\ t_1\ t_2 : \texttt{List}\ T_1} \quad \text{(T-Cons)}$$

$$\frac{\Gamma \vdash t_1 : \texttt{List}\ T_{11}}{\Gamma \vdash \texttt{isnil}[T_{11}]\ t_1 : \texttt{Bool}} \quad \text{(T-Isnil)}$$

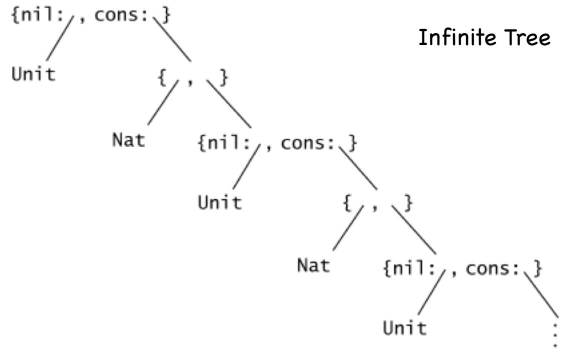
$$\frac{\Gamma \vdash t_1 : \texttt{List}\ T_{11}}{\Gamma \vdash \texttt{head}[T_{11}]\ t_1 : T_{11}} \quad \text{(T-Head)}$$

$$\frac{\Gamma \vdash t_1 : \texttt{List}\ T_{11}}{\Gamma \vdash \texttt{tail}[T_{11}]\ t_1 : \texttt{List}\ T_{11}} \quad \text{(T-Tail)}$$

Examples of Recursive Types

Lists

NatList = <nil:Unit, cons:{Nat, NatList}>



Infinite Tree

NatList = μX. <nil:Unit, cons:{Nat,X}>

This means that let NatList be the infinite type satisfying the equation:

X = <nil:Unit, cons:{Nat, X}>.

Defining functions over lists

- nil = <nil=unit> as NatList
- cons = λn:Nat. λl:NatList. <cons={n,l}> as NatList
- isnil = λl:NatList. case l of
<nil=u> ⇒ true
| <cons=p> ⇒ false;
- hd = λl:NatList. case l of <nil=u> ⇒ 0 | <cons=p> ⇒ p.1
- tl = λl:NatList. case l of <nil=u> ⇒ l | <cons=p> ⇒ p.2
- sumlist = fix (λs:NatList→Nat. λl:NatList.
if isnil l then 0 else plus (hd l) (s (tl l)))

Hungry Functions

- **Hungry Functions**: accepting any number of numeric arguments and always return a new function that is hungry for more

Hungry = μA. Nat→A

f : Hungry
f = fix (λf: Nat→Hungry. λn:Nat. f)

f 0 1 2 3 4 5 : Hugary

Streams

- **Streams**: consuming an arbitrary number of unit values, each time returning a pair of a number and a new stream

Stream = μA. Unit→ {Nat, A};

upfrom0 : Stream
upfrom0 = fix (λf: Nat→Stream. λn:Nat. λ_:Unit.
{n,f (succ n)}) 0;

hd : Stream → Nat
hd = λs:Stream. (s unit).1

(Process = μA. Nat→ {Nat, A})

20.1.2 EXERCISE [RECOMMENDED, ★★]: Define a stream that yields successive elements of the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, . . .). □

Objects

- **Objects**

```
Counter = μC. { get : Nat,
                inc : Unit→C,
                dec : Unit→C }
c : Counter
c = let create = fix (λf: {x:Nat}→Counter. λs: {x:Nat}.
                        { get = s.x,
                          inc = λ_:Unit. f {x=succ(s.x)},
                         dec = λ_:Unit. f {x=pred(s.x)} })
      in create {x=0};
```

((c.inc unit).inc unit).get ➔ 2

Recursive Values from Recursive Types

- **Recursive Values from Recursive Types**

  F = μA.A→T

  fixT = λf:T→T. (λx:(μA.A→T). f (x x))
  (λx:(μA.A→T). f (x x))

  (Breaking the strong normalizing property:
  diverge = λ_:Unit. fixT (λx:T. x) becomes typable)

Untyped Lambda Calculus

- **Untyped Lambda-Calculus**: we can embed the whole untyped lambda-calculus - in a well-typed way - into a statically typed language with recursive types.

  D= μX.X→X;

  lam : D
  lam = λf:D→D. f as D;

  ap : D
  ap = λf:D. λa:D. f a;

- Embedding

$$
\begin{array}{lcl}
x^{\star} & = & x \\
(\lambda x.M)^{\star} & = & \mathtt{lam}\ (\lambda x{:}D.\ M^{\star}) \\
(M\ N)^{\star} & = & \mathtt{ap}\ M^{\star}\ N^{\star}
\end{array}
$$

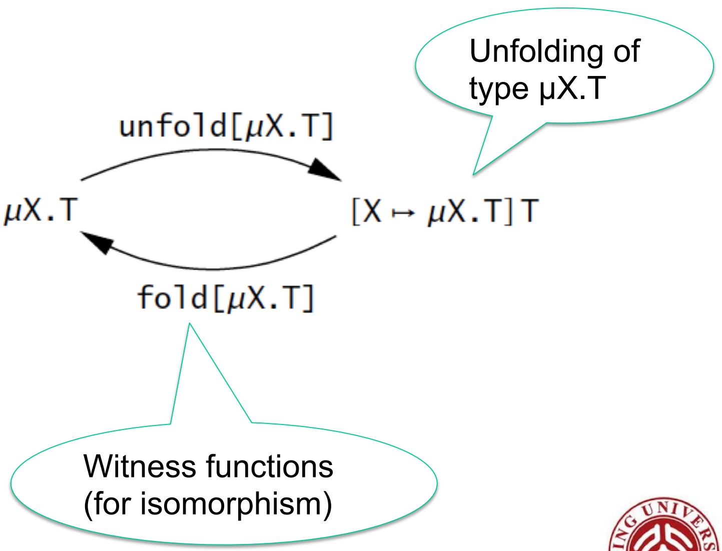
Formalities

What is the relation between the type $\mu X.T$ and its one-step unfolding?

Two Approaches

- The equi-recursive approach
  - takes these two type expressions as definitionally equal—interchangeable in all contexts— since they stand for the same infinite tree.
  - more intuitive, but places stronger demands on the typechecker.
- 2. The iso-recursive approach
  - takes a recursive type and its unfolding as different, but isomorphic.
  - Notationally heavier, requiring programs to be decorated with fold and unfold instructions wherever recursive types are used.

The Iso-Recursive Approach



Iso-recursive types (λμ)

→ μ *Extends λ→ (9-1)*

```
t ::= ...                          terms:
      fold [T] t                   folding
      unfold [T] t                 unfolding

v ::= ...                          values:
      fold [T] v                   folding

T ::= ...                          types:
      X                            type variable
      μX.T                         recursive type
```

New evaluation rules $\boxed{t \longrightarrow t'}$

$$\texttt{unfold [S] (fold [T] } v_1) \longrightarrow v_1 \quad \text{(E-UnfldFld)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{fold [T] } t_1 \longrightarrow \texttt{fold [T] } t_1'} \quad \text{(E-Fld)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{unfold [T] } t_1 \longrightarrow \texttt{unfold [T] } t_1'} \quad \text{(E-Unfld)}$$

New typing rules $\boxed{\Gamma \vdash t : T}$

$$\frac{U = \mu X.T_1 \qquad \Gamma \vdash t_1 : [X \mapsto U]T_1}{\Gamma \vdash \texttt{fold [U] } t_1 : U} \quad \text{(T-Fld)}$$

$$\frac{U = \mu X.T_1 \qquad \Gamma \vdash t_1 : U}{\Gamma \vdash \texttt{unfold [U] } t_1 : [X \mapsto U]T_1} \quad \text{(T-Unfld)}$$

Lists (Revisited)

NatList = μX. <nil:Unit, cons:{Nat,X}>

- 1-step unfolding of NatList:
  NLBody = <nil:Unit, cons:{Nat, NatList}>
- Definitions of functions on NatList
  - Constructors
    - nil = fold [NatList] (<nil=unit> as NLBody)
    - Cons = λn:Nat. λl:NatList.
      fold [NatList] <cons={n,l}> as NLBody
  - Destructors
    - hd = λl:NatList.
      case unfold [NatList] l of
      <nil=u> ⇒ 0
      | <cons=p> ⇒ p.1

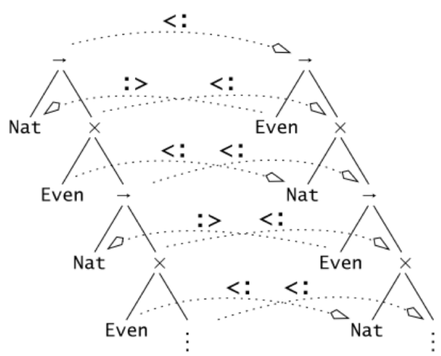
[Exercises: Define tl, isnil]

Subtyping

- Can we deduce

  μX. Nat → (Even × X) <: μX. Even→ (Nat × X)

  from Even <: Nat?



Homework

Problem (Chapter 20)

Natural number can be defined recursively by

$$Nat = \mu X. \langle zero: Nil, succ: X \rangle$$

Define the following functions in terms of fold and unfold.

(1) isZero n: check whether a natural number n is. zero or not.

(2) add1 n: increase a natural number n by 1.

(3) plus m n: add two natural numbers.