Chapter 6: Nameless Representation of Terms

Terms and Contexts

Shifting and Substitution

Bound Variables

- Recall: bound variables can be renamed, at any moment, to enable substitution:

$$
\begin{aligned}
[x \mapsto s]x &= s \\
[x \mapsto s]y &= y && \text{if } y \neq x \\
[x \mapsto s](\lambda y.t_1) &= \lambda y.\ [x \mapsto s]t_1 && \text{if } y \neq x \text{ and } y \notin FV(s) \\
[x \mapsto s](t_1\ t_2) &= [x \mapsto s]t_1\ [x \mapsto s]t_2
\end{aligned}
$$

- Variable Representation
  - Represent variables symbolically, with variable renaming mechanism
  - Represent variables symbolically, with bound variables are all different
  - "Canonically" represent variables in a way such that renaming is unnecessary
  - No use of variables: combinatory logic

Terms and Contexts

Nameless Terms

- **De Bruijin** Idea: Replacing named variables by natural numbers, where the number k stands for "the variable bound by the k'th enclosing λ".

  - Examples:

    λx.x   λ.0

    λx.λy. x (y x)   λ.λ. 1 (0 1).

- **Definition** [Terms]: Let T be the smallest family of sets $\{T_0, T_1, T_2, \ldots\}$ such that

  1. $k \in T_n$ whenever $0 \le k < n$;
  2. if $t_1 \in T_n$ and $n > 0$, then $\lambda.t_1 \in T_{n-1}$;
  3. if $t_1 \in T_n$ and $t_2 \in T_n$, then $(t_1\ t_2) \in T_n$.

  Note: T_n are set of terms with at most n free variables

Name Context

- **Naming Context**
  - To deal with terms containing free variables
  - Γ= x $\rightarrow$ 4; y $\rightarrow$ 3; z$\rightarrow$2; a $\rightarrow$ 1; b$\rightarrow$0

- **Examples**

  Under the naming context Γ, we have
  - x (y z)   4 (3 2)
  - λw. y w   λ. 4 0
  - λw.λa.x   λ.λ.6

Shifting and Subtitution

How to define substitution [k → s]t?

Shifting

- Under the naming context x → 1, z → 2

  [1 → 2 (λ.0)] λ.2 → ?

  i.e., [x → z (λw.w)] λy.x → ?

DEFINITION [SHIFTING]: The d-place shift of a term t above cutoff c, written $\uparrow_c^d (t)$, is defined as follows:

$$\uparrow_c^d (k) = \begin{cases} k & \text{if } k < c \\ k + d & \text{if } k \geq c \end{cases}$$

$$\uparrow_c^d (\lambda. t_1) = \lambda. \uparrow_{c+1}^d (t_1)$$

$$\uparrow_c^d (t_1\ t_2) = \uparrow_c^d (t_1)\ \uparrow_c^d (t_2)$$

We write $\uparrow^d (t)$ for $\uparrow_0^d (t)$. □

1. What is $\uparrow^2 (\lambda.\lambda.\ 1\ (0\ 2))$?

2. What is $\uparrow^2 (\lambda.\ 0\ 1\ (\lambda.\ 0\ 1\ 2))$?

Substitution

- Definition

$$[j \mapsto s]k \quad = \quad \begin{cases} s & \text{if } k = j \\ k & \text{otherwise} \end{cases}$$

$$[j \mapsto s](\lambda.t_1) \quad = \quad \lambda.\, [j+1 \mapsto \uparrow^1(s)]t_1$$

$$[j \mapsto s](t_1\ t_2) \quad = \quad ([j \mapsto s]t_1\ [j \mapsto s]t_2)$$

- Example

[1 → 2 (λ.0)] λ.2 ➜ λ.3 (λ.0)

i.e., [x → z (λw.w)] λy.x ➜ λy. z (λw.w)

Evaluation

$$(\lambda x.\ t_{12})\ t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

How to change the above rule for nameless representation?

Evaluation

$$(\lambda x.\, t_{12})\, t_2 \longrightarrow [x \mapsto t_2]t_{12},$$

$$(\lambda.\, t_{12})\, v_2 \longrightarrow \uparrow^{-1}([0 \mapsto \uparrow^1(v_2)]t_{12})$$

Example:

$$(\lambda.1\ 0\ 2)\ (\lambda.0) \longrightarrow 0\ (\lambda.0)\ 1$$

Homework

- Read Chapter 6.
- Do Exercise 6.2.5.

6.2.5   EXERCISE [★]: Convert the following uses of substitution to nameless form, assuming the global context is $\Gamma = $ a,b, and calculate their results using the above definition. Do the answers correspond to the original definition of substitution on ordinary terms from §5.3?

1. $[b \mapsto a]$ (b (λx.λy.b))

2. $[b \mapsto a\ (\lambda z.a)]$ (b (λx.b))

3. $[b \mapsto a]$ (λb. b a)

4. $[b \mapsto a]$ (λa. b a)   □