



软件科学基础

IndPrinciples: Induction Principles

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结构归纳法定理

Induction Principle

- Coq对每个递归定义的数据类型生成结构归纳法定理

```
Inductive nat : Type :=  
  | 0  
  | S (n : nat).
```

```
Check nat_ind :  
  forall P : nat -> Prop,  
    P 0 ->  
    (forall n : nat, P n -> P (S n)) ->  
    forall n : nat, P n.
```



结构归纳法定理的证明

- 直接按归纳定义展开，递归应用传入的证明即可

```
Print nat_ind.  
(*  
 * nat_ind =  
 * fun (P : nat -> Prop) (f : P 0)  
 *   (f0 : forall n : nat, P n -> P (S n)) =>  
 *   fix F (n : nat) : P n :=  
 *     match n with  
 *     | 0 => f  
 *     | S n0 => f0 n0 (F n0)  
 *   end  
 *)
```



更多结构归纳法定理

```
Inductive time : Type :=
  | day
  | night.
Check time_ind :
  forall P : time -> Prop,
    P day ->
    P night ->
    forall t : time, P t.
```

```
Inductive tree (X:Type) : Type :=
  | leaf (x : X)
  | node (t1 t2 : tree X).
Check tree_ind :
  forall (X : Type) (P : tree X -> Prop),
    (forall x : X, P (leaf X x)) ->
    (forall t1 : tree X,
      P t1 -> forall t2 : tree X, P t2 -> P (node X t1 t2)) ->
    forall t : tree X, P t.
```



参数的位置影响 结构归纳法定理

```
Inductive tree' : Type -> Type :=  
  | leaf' (X: Type) (x : X): tree' X  
  | node' (X: Type) (t1 t2 : tree' X) : tree' X.
```

```
Check tree'_ind :  
forall P : forall T : Type, tree' T -> Prop,  
(forall (X : Type) (x : X), P X (leaf' X x)) ->  
(forall (X : Type) (t1 : tree' X),  
P X t1 -> forall t2 : tree' X, P X t2 -> P X (node' X t1 t2)) ->  
forall (T : Type) (t : tree' T), P T t.
```

右边的参数不在定理一开始传入，对应P的形式也发生变化



induction

Theorem `plus_n_0` : forall n:nat, n = n + 0.

Proof.

```
intros n. induction n as [| n' IHn']. Show Proof.
```

```
(* (fun n : nat =>
  nat_ind (fun n0 : nat => n0 = n0 + 0) ?Goal
    (fun (n' : nat) (IHn' : n' = n' + 0) => ?Goal0) n) *)
```

Print `nat_ind`.

```
(* nat_ind : forall P : nat -> Prop,
  P 0 -> (forall n : nat, P n -> P (S n)) -> forall n : nat, P n *)
- (* n = 0 *) reflexivity.
- (* n = S n' *) simpl. rewrite <- IHn'. reflexivity. Qed.
```

induction: 应用结构归纳法定理



命题对应的结构归纳法定理

根据一般归纳类型的结构归纳法定理猜测：

```
Inductive ev : nat -> Prop :=  
| ev_0 : ev 0  
| ev_SS (n : nat) (H : ev n) : ev (S (S n)).
```

```
ev_ind :  
  forall P : forall (n:nat), ev n -> Prop,  
    P 0 ev_0 ->  
    (forall (n:nat) (E:ev n), P n E ->  
     P (S (S n)) (ev_SS n E))->  
    forall (n':nat) (E':ev n), P n' E'.
```

P的第二个参数（即ev n的证明）没有用，因为Coq只允许从证明构建证明，实践中也不会有函数或谓词作用于证明上。



尝试从证明返回别的值

```
Fail Definition strange (H:forall n, ev n) (m:nat) :=  
  match H m with  
  | ev_0 => 0  
  | ev_SS _ _ => S 0  
  end.
```

The command has indeed failed with message:
Incorrect elimination of "H m" in the inductive type "ev":
the return type has sort "Set" while it should be "SProp" or "Prop".
Elimination of an inductive object of sort Prop
is not allowed on a predicate in sort Set
because **proofs can be eliminated only to build proofs.**

提问:

- 什么时候我们会输入命题，输出命题？
- 什么时候我们会输入证明，输出证明？
- 什么时候我们会输入命题，输出证明？



命题对应的结构归纳法定理

```
Check ev_ind :  
  forall P : nat -> Prop,  
    P 0 ->  
    (forall n : nat, ev n -> P n -> P (S (S n))) ->  
    forall n : nat, ev n -> P n.
```

去掉命题本身的证明作为参数
P只作用在ev的参数上



命题对应定理的证明

```
Print ev_ind.  
(* ev_ind =  
fun (P : nat -> Prop) (f : P 0)  
  (f0 : forall n : nat, ev n -> P n -> P (S (S n))) =>  
fix F (n : nat) (e : ev n) {struct e} : P n :=  
  match e with  
  | ev_0 => f  
  | ev_SS n0 H => f0 n0 H (F n0 H)  
end *)
```

和之前类似，展开 $ev\ n$ 的归纳定义，并递归传入参数



关系对应的结构归纳法定理

```
Inductive le : nat -> nat -> Prop :=  
  | le_n (n : nat) : le n n  
  | le_S (n m : nat) (H : le n m) : le n (S m).
```

```
Check le_ind : forall P : nat -> nat -> Prop,  
  (forall n : nat, P n n) ->  
  (forall n m : nat, le n m -> P n m -> P n (S m)) ->  
  forall n n0 : nat, le n n0 -> P n n0.
```

这是课本上定义的le，对应归纳法定理中P作用在le的所有参数上



参数位置的影响

Print le.

```
(* ===> Inductive le (n : nat) : nat -> Prop :=  
      le_n : n <= n  
      | le_S : forall m : nat, n <= m -> n <= S m *)
```

Check le_ind:

```
forall (n : nat) (P : nat -> Prop),  
  P n ->  
  (forall m : nat, n <= m -> P m -> P (S m)) ->  
  forall n0 : nat, n <= n0 -> P n0.
```

n作为整体定理的参数在一开始传入，对应P也少了一个参数
这是标准库中的le，对应归纳法定理中P只作用在index上，归纳形式更简单



eq_ind定理

```
Inductive eq (A : Type) (x : A) : A -> Prop
:= eq_refl : x = x
```

```
Check eq_ind :
  forall (A: Type) (x:A) (P : A -> Prop),
    P x -> forall y : A, x=y -> P y.
```



逻辑非:discriminate

```
Theorem zero_not_one : 1 <> 0.  
Proof.  
  intros contra.  
  discriminate contra.  
Qed.
```

```
Definition zero_not_one' : 0 <> 1 :=  
  fun contra : 0 = 1 => eq_ind 0  
    (fun e:nat => match e with  
      | 0 => True  
      | S _ => False  
    end) I 1 contra.
```

```
eq_ind : forall (A : Type) (x : A) (P : A -> Prop),  
  P x -> forall y : A, x = y -> P y
```

discriminate: 基于两个不同Constructor相等的证明构造False的证明



rewrite

```
Theorem plus_id_example : forall n m:nat,  
  n=m -> n+n=m+m.
```

Proof.

```
intros n m H.  
rewrite <- H.  
reflexivity.
```

Qed.

```
Definition plus_id_example' : forall n m:nat,  
  n=m -> n+n=m+m :=  
fun (n m : nat) (H : n=m) =>  
eq_ind n (fun (m:nat) => n+n=m+m) eq_refl m H.
```

rewrite: 利用eq_ind实现相等内容的替换



作业

- 完成IndPrinciples中standard非optional的3道习题
 - 请使用最新英文版教材